

TRANSITIVELY-SATURATED PROPERTY, RECURRENCE AND LYAPUNOV REGULARITY

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Abstract. The topological entropy of various gap-sets on recurrence and Birkhoff regularity were considered in [82]. In this paper we aim to generalize the results of [82] in following ways:

- (1) Birkhoff regularity is generalized to Lyapunov version: Birkhoff ergodic average of a continuous function (the case additive case) is replaced by Lyapunov exponents of asymptotically additive functions (in particular, the case to study level sets of Lyapunov exponents can be as a fine generalization of [32]);
- (2) All considered recurrences are restricted on transitive points and in present paper we also consider another kind of recurrence, called Banach recurrence;
- (3) Moreover, the requirements on dynamical systems are weakened: the density assumption on periodic orbits is weakly replaced by a measure with full support.

In this process, we strengthen Pfister and Sullivan's result of [65] from saturated property to transitively-saturated property (and from single-saturated property to transitively-convex-saturated property).

1. INTRODUCTION

In this paper, a dynamical system (X, T) means always that X is a compact metric space and $T : X \rightarrow X$ is a continuous map.

Definition 1.1. For a collection of subsets $Z_1, Z_2, \dots, Z_k \subseteq X$ ($k \geq 2$), we say $\{Z_i\}$ has *full entropy gaps* with respect to $Y \subseteq X$ (simply, FEG w.r.t Y) if

$$h_{top}(T, (Z_{i+1} \setminus Z_i) \cap Y) = h_{top}(T, Y) \quad \text{for all } 1 \leq i < k,$$

where $h_{top}(T, Z)$ denotes the topological entropy of a set $Z \subseteq X$.

Often, but not always, the sets Z_i are nested ($Z_i \subseteq Z_{i+1}$). Note that for any system with zero topological entropy, it is obvious that any collection $\{Z_i\}$ has full entropy gaps with respect to any $Y \subseteq X$. In present paper, we always assume that

T is not uniquely ergodic, not minimal and $h_{top}(T) > 0$.

Note that positive entropy implies that X is a compact metric space with infinitely many points. We refer to [83] for ergodic theory and to [3, 62] for dimension theory.

Given $x \in X$, let $\omega_T(x)$ denote the ω -limit set of x , let M_x be the limit set of the empirical measures for x and let $C_x := \overline{\bigcup_{\nu \in M_x} S_\nu}$ where S_ν denotes the support of measure ν . In this paper, we consider following subsets of X according

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to different asymptotic behavior:

$$\begin{aligned}
Per &:= \{ \text{periodic points of } T \}, \\
A &:= \{ \text{almost periodic points of } T \} = \{ \text{points contained in minimal set} \}, \\
Rec &:= \{ \text{recurrent points of } T \}, \\
BR &:= \{ \text{Banach recurrent points} \}, \\
\Omega &:= \{ \text{non-wandering points of } T \}, \\
Tran &:= \{ x \in X \mid \omega_T(x) = X \}, \\
W &:= \{ x \in Rec \mid S_\mu = C_x \text{ for every } \mu \in M_x \}, \\
QW &:= \{ x \in Rec \mid C_x = \omega_T(x) \}, \\
V &:= \{ x \in QW \mid \exists \mu \in M_x \text{ such that } S_\mu = C_x \}.
\end{aligned}$$

The notions of periodic, recurrent and non-wandering can be found in [83], the notion of almost periodic or minimal can be seen in [11, 40, 38, 39, 55] and others, for example, see [88, 87, 89, 90]. We will recall their definitions, equivalent statements and relations later. Such sets are all T -invariant and they satisfy $Per \subseteq A \subseteq W \subseteq V \subseteq QW \subseteq Rec \subseteq \Omega$ and $Tran \subseteq Rec$.

A point $x \in X$ is generic for some invariant measure μ means that $M_x = \{\mu\}$ (or equivalently, Birkhoff averages of all continuous functions converge to the integral of μ). Let G_μ denote the set of all generic points for μ . Let $M(X)$, $M(T, X)$, $M_{erg}(T, X)$, $M_p(T, X)$ and $M_{min}(T, X)$ denote the set of all probability measures, T -invariant measures, T -ergodic measures T -periodic measures and T -invariant measures supported on minimal sets (i.e., T -invariant measures whose support should be minimal) respectively. We also consider

$$\begin{aligned}
QR &:= \{ \text{quasiregular points of } T \} = \cup_{\mu \in M(T, X)} G_\mu, \\
I &:= \{ \text{irregular points of } T \} = X \setminus QR, \\
QR_{erg} &:= \{ \text{points generic for ergodic measures} \} = \cup_{\mu \in M_{erg}(T, X)} G_\mu, \\
QR_d &:= \{ \text{points of density in } QR \} = \cup_{\mu \in M(T, X)} (G_\mu \cap S_\mu), \\
R &:= \{ \text{regular points of } T \} = QR_d \cap QR_{erg} = \cup_{\mu \in M_{erg}(T, X)} (G_\mu \cap S_\mu).
\end{aligned}$$

Such sets are all T -invariant and remark that

$$R \subseteq QR_d \cup QR_{erg} \subseteq QR = X \setminus I.$$

Most notions except irregular point are from [60] (for quasiregular point, also see [25]) and the notion of irregular point can be found in [63, 3, 77, 7] etc.

For any compact connected $K \subseteq M(T, X)$, let $G_K = \{x \in X \mid M_x = K\}$, $G_K^T = \{x \in Tran \mid M_x = K\}$. They are saturated set of K and transitively-saturated set of K respectively.

Definition 1.2. We say that the system T has *saturated* property or T is *saturated* (simply, S), if for any compact connected nonempty set $K \subseteq M(T, X)$,

$$(1.1) \quad h_{top}(T, G_K) = \inf\{h_\mu(T) \mid \mu \in K\}.$$

We say that the system T has *locally-saturated* property or T is *locally-saturated* (simply, LS), if for any compact connected nonempty set $K \subseteq M(T, X)$, any nonempty open set $U \subseteq X$,

$$(1.2) \quad h_{top}(T, G_K \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}.$$

In parallel, one can define *transitively-saturated* (simply, TS) and *locally-transitively-saturated* (simply, LTS), just replacing G_K by G_K^T in (1.1) and (1.2). On the other hand, (locally-)transitively-, transitively-, locally-)single-saturated means (1.1) holds when K is a singleton. If above equalities hold for the K that consists of convex sum

of two invariant measures, we say the system T is convex-saturated, locally-convex-saturated, transitively-convex-saturated, locally-transitively-convex-saturated, respectively.

Remark 1.3. From [15, Theorem 3] we know for any dynamical system (X, T) , if K is a singleton only with one ergodic measure μ , then one has naturally

$$(1.3) \quad h_{top}(T, G_K) = h_\mu(f).$$

However, it does not imply $h_{top}(T, G_K^T) = h_\mu(f)$ except $S_\mu = X$. For example, if T is a product map by identity map on $[0, 1]$ and a system with positive topological entropy, then for any $K = \{\mu\}$ that μ is ergodic with positive metric entropy, G_K^T is empty so that $h_{top}(T, G_K^T) = 0 < h_\mu(f)$.

Note that locally-saturated is stronger than saturated (just taking $U = X$), saturated is stronger than convex-saturated and the later is stronger than single-saturated. We will give a basic discussion on the relations of these concepts in Lemma 3.1 (see below).

Definition 1.4. We say T has *entropy-dense* property, if for any $\nu \in M(T, X)$, any neighborhood $G \subseteq M(X)$ of ν and any $h_* < h_\nu(T)$, there exists an ergodic measure $\mu \in G \cap M(T, X)$ such that $S_\mu \neq X$ and $h_\mu(T) > h_*$.

Let us recall the concept of asymptotically additive introduced from [32], which helps us to study multifractal behavior of Lyapunov exponents.

Definition 1.5. A sequence of functions $\phi_n : X \rightarrow R$ is said to be asymptotically additive if for each $\epsilon > 0$, there exists a continuous function $\phi : X \rightarrow R$ such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in X} |\phi_n(x) - S_n(x)| \leq \epsilon,$$

where $S_n = \sum_{k=0}^{n-1} \phi \circ T^k$.

Given a sequence of functions $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$, the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \phi_n(x)$ (if exists) is called *the Lyapunov exponent of Φ at x* , see [32]. let

$$R_\Phi := \{x \in X \mid \text{averages } \frac{1}{n} \phi_n(x) \text{ converge as } n \rightarrow +\infty\}.$$

For convenience, we call R_Φ to be *Lyapunov- Φ -regular set* (simply, Φ -regular set). Define the Φ -irregular set (or Lyapunov- Φ -irregular set) $I_\Phi := X \setminus R_\Phi$. For any $a \in \mathbb{R}$, define level set of Lyapunov exponents

$$R_{\Phi, a} := \{x \in X \mid \lim_{n \rightarrow \infty} \frac{1}{n} \phi_n(x) = a\}.$$

Remark that $R_\Phi = \bigsqcup_{a \in \mathbb{R}} R_{\Phi, a}$, where \bigsqcup denotes disjoint union. There are lots of classical results for $R_{\Phi, a}$, for example, see [4, 32] (see [65, 78, 59] for additive functions). Define the domain of the multifractal spectrum for ergodic averages of Φ (called Lyapunov spectrum of Φ for convenience),

$$L_\Phi := [\inf_{\mu \in M(T, X)} \chi_\Phi(\mu), \sup_{\mu \in M(T, X)} \chi_\Phi(\mu)],$$

where $\chi_\Phi(\mu) := \liminf_{n \rightarrow \infty} \int \frac{1}{n} \phi_n d\mu$. Let $\text{Int}(L_\Phi)$ denote the interior of L_Φ .

1.1. Main Results. Now we start to state our main theorem.

Theorem A. *Suppose that (X, T) is a saturated dynamical system with entropy-dense property and there is an invariant measure with full support. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. Then*

- (1) $\{A \cup R, QR, W, V, QW\}$ has FEG w.r.t. Tran .
- (2) $\{A \cup R, QR, W, V, QW\}$ has FEG w.r.t. $\text{Tran} \cap R_\Phi$.
- (3) If $I_\Phi \neq \emptyset$, then $\{QR, W, V, QW\}$ has FEG w.r.t. $I_\Phi \cap \text{Tran}$.
- (4) If $I_\Phi \neq \emptyset$, then for any $a \in \text{Int}(L_\Phi)$, $\{A \cup R, QR, W, V, QW\}$ has FEG w.r.t. $R_{\Phi,a} \cap \text{Tran}$.
- (5) If further T is transitively-convex-saturated, then $\{QW, I \cap BR\}$ can also be stated to have FEG w.r.t. $X, R_\Phi, I_\Phi, R_{\Phi,a}$ restricted on transitive points.
- (6) If the set $\{\omega \in M(T, X) \mid S_\omega \text{ is minimal}\}$ is dense in $M(T, X)$, then $\{QW, I\}$ can be stated to have FEG w.r.t. $X, R_\Phi, I_\Phi, R_{\Phi,a}$.
- (7) If the assumption of saturated property is weakly replaced by convex-saturated, then items (1)-(4) and item (6) still hold except the part for $\{V, QW\}$.

Remark 1.6. Since there is an invariant measure with full support, the considered system T in Theorem A is transitive, see Lemma 3.8 below. Thus T is an E-system (Recall that E-system means a transitive dynamical system with an invariant measure with full support) and $\Omega = X = \overline{\text{Rec}}$. If further $\{\mu \in M(T, X) \mid S_\mu \text{ is minimal}\}$ is dense in $M(T, X)$ (for example, T satisfies specification or g -almost product property), then almost periodic set A is dense in X (by Lemma 3.11 below) so that T is also a M-system (Recall that M-system means a transitive dynamical system that almost periodic set A is dense in X). However, the dynamical system considered in [82] is required density of periodic points and periodic measures and hence is a P-system (Recall that P-system means a transitive dynamical system that periodic set Per is dense in X). Notice that P-system is stronger than M-system and the later is stronger than E-system, thus the assumptions in Theorem A is weaker than ones in [82].

We will give the proof of Theorem A in Section 6.1.

Theorem B. *Suppose that (X, T) satisfies g -almost product property and there is an invariant measure with full support. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. Then T is locally-transitively-convex-saturated, the set $\{\omega \in M(T, X) \mid S_\omega \text{ is minimal}\}$ is dense in $M(T, X)$, and thus all items of Theorem A hold except the part $\{V, QW\}$. If further T satisfies uniform separation, then T is locally-transitively-saturated and all items of Theorem A hold.*

Remark 1.7. (i). On one hand, Theorem B generalizes Pfister and Sullivan's result of [65] from saturated property to transitively-saturated property (and from single-saturated property to transitively-convex-saturated property). On the other hand, Theorem B generalizes [82, Theorem 1.2, 1.3, 1.4, 8.6] in following ways: (1) Birkhoff regularity is generalized to Lyapunov version: Birkhoff ergodic average of a continuous function (the case additive case) is replaced by Lyapunov exponents of asymptotically additive functions (in particular, the case to study level sets of Lyapunov exponents can be as a fine generalization of [32]); (2) All considered recurrences are restricted on transitive points and in present paper we also consider another kind of recurrence, called Banach recurrence; (3) Moreover, the requirements on dynamical systems are weakened: the density assumption on periodic orbits is weakly replaced by a measure with full support. In particular, Theorem B is a generalization of [82, Theorem 1.6], for which the assumption of Bowen's specification is weakened by g -almost product property and existence of a measure with full support.

(ii). The first part of Theorem B gives a partial positive answer of [82, Question 9.8] which asked related results for systems with specification or g -almost product property but without uniform separation. Recall from [81] that for any system with almost specification (an analogue of g -almost product property), every ϕ -irregular set either is empty or carries full topological entropy. So the first part of Theorem B on irregular discussion is a fine generalization of [81]. For the part $\{V, QW\}$, it is still unknown.

From [64, Theorem 2.1] or [82, Lemma 2.10] we know g -almost product property implies entropy-dense. By Lemma 3.1 (see below), locally-transitively-saturated are equivalent to transitively-saturated and they are stronger than saturated property. Thus, in order to prove Theorem B, by Theorem A we only need to prove following.

Proposition A. *Suppose that (X, T) satisfies g -almost product property and there is an invariant measure with full support. Then T is transitively-convex-saturated, the set $\{\omega \in M(T, X) \mid \mu \text{ is ergodic, } S_\omega \text{ is minimal}\}$ is dense in $M(T, X)$ and almost periodic set A is dense in X . If further T has uniform separation, then T is transitively-saturated.*

Proposition A generalizes Pfister and Sullivan's result of [65] from saturated property to transitively-saturated property. We will give the proof of Proposition A in Section 6.2.

From [24], we know that for any dynamical system with specification property (not necessarily Bowen's strong version), the almost periodic points are dense in X and the invariant measures supported on minimal sets are dense in the space of invariant measures. By Lemma 3.9 (see below) there is some invariant measure with full support. From [65, Proposition 2.1] we know specification implies g -almost product property so that the assumptions of g -almost product property and existence of a measure with full support in Theorem B can be replaced by specification. That is,

Corollary A. *Suppose that (X, T) satisfies specification property. Then all results of Theorem B hold except the part $\{V, QW\}$. If further T has uniform separation property. Then all results of Theorem B hold.*

This is a generalization of [82, Theorem 1.6], replacing the assumption of Bowen's specification by specification.

1.2. Examples.

1.2.1. *Subshifts of finite type and hyperbolic systems.* The results of Corollary A are suitable for

- (1) all topological mixing subshifts of finite type and soft subshifts (in particular, all full shifts on finite alphabets);
- (2) all subsystems restricted on topological mixing locally maximal expanding set or hyperbolic set (in particular, all topological mixing expanding maps or topological mixing hyperbolic diffeomorphisms, called Anosov).

This is because such examples all satisfy Bowen's specification (see [16, 86] for item (1) and see [13] for item (2), also cf. [25]) and are expansive which is stronger than uniform separation, see [65].

1.2.2. *Factors.* From [25, Proposition 21.4] we know that the factor of a system with Bowen's specification has Bowen's specification, and thus Corollary A applies except $QW \setminus V$.

On the other hand, from [25, Proposition 16.8, item a)] we know expansiveness is invariant under conjugacy. So a system conjugated to a system with specification and expansiveness is also expansive and has specification and thus Corollary A applies.

1.2.3. Non-hyperbolic systems. The results of this paper are also applicable to some dynamical systems beyond uniform hyperbolicity. From [37] we know that non-hyperbolic diffeomorphism f with C^{1+Lip} smoothness, conjugated to a transitive Anosov diffeomorphism g , exists even the conjugation and its inverse is Hölder continuous. From Section 1.2.1 we know g is expansive and has Bowen's specification. Thus from Section 1.2.2 we get that the non-hyperbolic system f satisfies specification and expansiveness so that Corollary A applies.

1.2.4. Time- t map of Anosov flow. The results of Corollary A can also be applicable to certain dynamical systems not necessarily having periodic orbits, such as time-1 maps of all transitive Anosov flows.

Corollary B. *Let T be the time-1 map of a transitive Anosov flow. Then all results of Theorem B hold.*

This is a further step of [82, Theorem 7.14]. Let us explain why Corollary B holds. From Section 4.3 of [80] we know the time-1 map of a transitive Anosov flow satisfies specification. In this case, f is partially hyperbolic with one-dimension central bundle. Then f is far from tangency so that f is entropy-expansive (see [53] or see [26, 61]). Recall that from [56] entropy-expansive implies asymptotically h -expansive and from Theorem 3.1 of [65] any expansive or asymptotically h -expansive system satisfies uniform separation property. So Corollary A applies.

We also learned from [2] in the class of all solenoidal automorphisms, the class of automorphisms with specification is wider than the class of automorphisms with Bowen's specification. So Corollary A has more applications.

1.2.5. β -shifts.

Corollary C. *All results of Theorem B hold for any β -shift.*

This is a further generalization of [82, Theorem 1.9]. Let us explain why Corollary C holds. We know that any β -shift is expansive (which is stronger than uniform separation, see [65]) and from [65] it always satisfies g -almost product property. Furthermore, from [74] we know that periodic points are dense in the whole space and the periodic measures are dense in the space of invariant measures (i.e., $\overline{M_p(T, X)} = M(T, X)$). By Lemma 3.9 (see below) there is some invariant measure with full support. Thus Theorem B applies.

It is worth mentioning that from [21] the set of parameters of β for which specification holds, is dense in $(1, +\infty)$ but has Lebesgue zero measure.

1.2.6. Interval maps. It is known from [12, 21] that any topologically mixing interval map satisfies Bowen's specification but maybe not have uniform separation. For example, Jakobson [48] showed that there exists a set of parameter values $\Lambda \subseteq [0, 4]$ of positive Lebesgue measure such that if $\lambda \in \Lambda$, then the logistic map $f_\lambda(x) = \lambda x(1 - x)$ is topologically mixing. By Corollary A, we can deduce that the results of Theorem A hold for mixing interval maps except considering the gap-set $QW \setminus V$.

1.2.7. *Dynamics with specification but without uniform separation.* In [65], Pfister and Sullivan gave an example of a dynamical system with finite topological entropy, for which the entropy density of ergodic measures is true (specification property is true), but the uniform separation property and the upper semi-continuity of the entropy map fail. This example is a subshift of the shift space $Y := [-1, 1]^{\mathbb{Z}^+}$, see [65, Page 952-Page 953] for more details. By Corollary A, we can deduce that the results of Theorem A hold for this example except considering the gap-set $QW \setminus V$.

1.3. **Multifractal analysis of Lyapunov exponents.** The concept of asymptotically sub-additive potentials of [32] is mainly motivated by some works on the Lyapunov exponents of matrix products [33, 34, 35] and the Lyapunov exponents of differential maps on nonconformal repellers [5]. As a fine generalization of such works, our Theorem A applies to matrix products in [33, 34, 35] and nonconformal repellers in [5].

1.4. **Organization of this paper.** In Section 2 and Section 3 we recall some notions and introduce some useful lemmas. In Section 4 and Section 5 we will study gap-sets w.r.t. irregular set and level set respectively. In Section 6 we complete the proof of main results.

2. PRELIMINARIES

2.1. **Invariant measures.** Let $\{\varphi_j\}_{j \in \mathbb{N}}$ be a dense subset of $C(X, \mathbb{R})$ which is the space of continuous functions, then

$$d(\xi, \tau) = \sum_{j=1}^{\infty} \frac{|\int \varphi_j d\xi - \int \varphi_j d\tau|}{2^{j+1} \|\varphi_j\|}$$

defines a metric on $M(X)$ for the *weak** topology [83], where

$$\|\varphi_i\| = \max\{|\varphi_i(x)| : x \in X\}.$$

Note that

$$(2.1) \quad d(\xi, \tau) \leq 1 \text{ for any } \xi, \tau \in M(X).$$

2.2. **Entropy.** Let $T : X \rightarrow X$ be a continuous map of a compact metric space X . Now let us to recall the definition of topological entropy in [15] by Bowen.

Let $x \in X$. The dynamical ball $B_n(x, \varepsilon)$ is the set

$$B_n(x, \varepsilon) := \{y \in X \mid \max\{d(T^j(x), T^j(y)) \mid 0 \leq j \leq n-1\} \leq \varepsilon\}.$$

Let $E \subseteq X$, and $\mathfrak{F}_n(E, \varepsilon)$ be the collection of all finite or countable covers of E by sets of the form $B_m(x, \varepsilon)$ with $m \geq n$. We set

$$C(E; t, n, \varepsilon, T) := \inf \left\{ \sum_{B_m(x, \varepsilon) \in \mathcal{C}} 2^{-tm} : \mathcal{C} \in \mathfrak{F}_n(E, \varepsilon) \right\},$$

and

$$C(E; t, \varepsilon, T) := \lim_{n \rightarrow \infty} C(E; t, n, \varepsilon, T).$$

Then

$$h_{top}(E, \varepsilon, T) := \inf\{t : C(E; t, \varepsilon, T) = 0\} = \sup\{t : C(E; t, \varepsilon, T) = \infty\}$$

and the *topological entropy* of E is defined as

$$h_{top}(T, E) := \lim_{\varepsilon \rightarrow 0} h_{top}(E, \varepsilon, T).$$

In particular, if $E = X$, we also denote $h_{top}(T, X)$ by $h_{top}(T)$. It is known from [15] that if E is an invariant compact subset, then the topological entropy $h_{top}(T, E)$ is same as the classical definition (for classical definition of topological entropy, see Chapter 7 in [83]).

Let us recall some basic facts about topological entropy. From [15] we know for any $Y \subseteq X$,

$$(2.2) \quad h_{top}(T, fY) = h_{top}(T, Y).$$

and for any subsets $Y_1 \subseteq Y_2 \subseteq X$,

$$(2.3) \quad h_{top}(T, Y_1) \leq h_{top}(T, Y_2).$$

If one considers a collection of subsets of X : $\{Y_i\}_{i=1}^{+\infty}$, from [15] we know that the topological entropy satisfies

$$(2.4) \quad h_{top}(T, \bigcup_{i=1}^{+\infty} Y_i) = \sup_{i \geq 1} h_{top}(T, Y_i).$$

Let $\xi = \{V_i | i = 1, 2, \dots, k\}$, be a finite partition of measurable sets of X . The entropy of a probability measure $\nu \in M(X)$ with respect to ξ is

$$H(\nu, \xi) := - \sum_{V_i \in \xi} \nu(V_i) \log \nu(V_i).$$

We write $T^{\vee n} \xi := \bigvee_{k \in \Lambda} T^{-k} \xi$. The entropy of an invariant measure $\nu \in M(T, X)$ with respect to ξ is

$$h(T, \nu, \xi) := \lim_{n \rightarrow \infty} \frac{1}{n} H(\nu, T^{\vee n} \xi),$$

and the *metric entropy* of ν is

$$h_\nu(T) := \sup_{\xi} h(T, \nu, \xi).$$

More information of metric entropy, see Chapter 4 of [83].

For convenience, we write h_{top} and h_ω to denote $h_{top}(T)$ and $h_\omega(T)$.

2.3. Specification property and Product property. Firstly we recall the definition of specification property which is stronger than g -almost product property, see [25, 72, 18, 19, 13, 80]. Let T be a continuous map of a compact metric space X .

Definition 2.1. We say that the dynamical system T satisfies *specification property*, if the following holds: for any $\epsilon > 0$ there exists an integer $M(\epsilon)$ such that for any $k \geq 2$, any k points x_1, \dots, x_k , any integers

$$a_1 \leq b_1 < a_2 \leq b_2 \cdots < a_k \leq b_k$$

with $a_{i+1} - b_i \geq M(\epsilon)$ ($1 \leq i \leq k-1$), there exists a point $x \in X$ such that

$$(2.5) \quad d(T^j(x), T^j(x_i)) \leq \epsilon, \quad \text{for } a_i \leq j \leq b_i, 1 \leq i \leq k.$$

$$(2.6) \quad \text{In other words, the set } \hat{B} = \bigcap_{i=1}^k f^{-a_i} B_{b_i - a_i}(f^{a_i} x_i, \epsilon) \text{ is nonempty.}$$

The original definition of specification, due to Bowen, was stronger.

Definition 2.2. We say that the dynamical system T satisfies *Bowen's specification property*, if under the assumptions of Definition 2.1 and for any integer $p \geq M(\epsilon) + b_k - a_1$, there exists a point $x \in X$ with $T^p(x) = x$ satisfying (2.5).

Now we start to recall the concept g -almost product property in [65] (there is a slightly weaker variant, called almost specification, see [81]). It is weaker than specification property (see Proposition 2.1 in [65]). A striking and typical example of g -almost product property (and almost specification) is that it applies to every β -shift [65, 81]. In sharp contrast, the set of β for which the β -shift has specification property has zero Lebesgue measure [21, 75].

Let $\Lambda_n = \{0, 1, 2, \dots, n-1\}$. The cardinality of a finite set Λ is denoted by $\#\Lambda$. Let $x \in X$. The dynamical ball $B_n(x, \varepsilon)$ is the set

$$B_n(x, \varepsilon) := \{y \in X \mid \max\{d(T^j(x), T^j(y)) \mid j \in \Lambda_n\} \leq \varepsilon\}.$$

Definition 2.3. Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a given nondecreasing unbounded map with the properties

$$g(n) < n \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{g(n)}{n} = 0.$$

The function g is called *blowup function*. Let $x \in X$ and $\varepsilon > 0$. The g -blowup of $B_n(x, \varepsilon)$ is the closed set $B_n(g; x, \varepsilon) :=$

$$\{y \in X \mid \exists \Lambda \subseteq \Lambda_n, \#(\Lambda_n \setminus \Lambda) \leq g(n) \quad \text{and} \quad \max\{d(T^j(x), T^j(y)) \mid j \in \Lambda\} \leq \varepsilon\}.$$

Definition 2.4. We say that the dynamical system T satisfies *g -almost product property* with blowup function g , if there is a nonincreasing function $m : \mathbb{R}^+ \rightarrow \mathbb{N}$, such that for any $k \geq 2$, any k points $x_1, \dots, x_k \in X$, any positive $\varepsilon_1, \dots, \varepsilon_k$ and any integers $n_1 \geq m(\varepsilon_1), \dots, n_k \geq m(\varepsilon_k)$,

$$\bigcap_{j=1}^k T^{-M_{j-1}} B_{n_j}(g; x_j, \varepsilon_j) \neq \emptyset,$$

where $M_0 := 0, M_i := n_1 + \dots + n_i, i = 1, 2, \dots, k-1$.

It is well known that the natural projection $x \mapsto \delta_x$ is continuous and if we define operator T_f on $M(X)$ by formula $T_f(\mu)(A) = \mu(f^{-1}(A))$, then we can identify (X, f) with T_f restricted to the set of Dirac measures (these systems are conjugate). Therefore, without loss of generality we will assume that $d(x, y) = d(\delta_x, \delta_y)$. Denote a ball in $M(X)$ by

$$\mathcal{B}(\nu, \zeta) := \{\mu \in M(X) : d(\nu, \mu) \leq \zeta\}.$$

Lemma 2.1. [65, Lemma 2.1] Assume that (X, T) satisfies g -almost product property. Let $x_1, \dots, x_k \in X$, $\varepsilon_1, \dots, \varepsilon_k$ and $q_1 \geq m(\varepsilon_1), \dots, q_k \geq m(\varepsilon_k)$ be given. Assume that

$$\Upsilon_{q_j}(x_j) \in \mathcal{B}(\nu_j, \zeta_j), \quad j = 1, 2, \dots, k.$$

Then for any $z \in \bigcap_{j=1}^k T^{-Q_{j-1}} B_{q_j}(g; x_j, \varepsilon_j)$ and any probability measure α

$$d(\Upsilon_{Q_k}(z), \alpha) \leq \sum_{j=1}^k \frac{n_j}{Q_k} (\zeta_j + \varepsilon_j + \frac{g(q_j)}{q_j})$$

where $Q_0 = 0, Q_i = q_1 + \dots + q_i$.

2.4. Uniform separation. Now we recall the definition of uniform separation property [65]. For $x \in X$, define

$$\Upsilon_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)}$$

where δ_y is the Dirac probability measure supported at $y \in X$. For $\delta > 0$ and $\varepsilon > 0$, two points x and y are (δ, n, ε) -separated if

$$\#\{j : d(T^j x, T^j y) > \varepsilon, j \in \Lambda_n\} \geq \delta n.$$

A subset E is (δ, n, ε) -separated if any pair of different points of E are (δ, n, ε) -separated. Let $F \subseteq M(X)$ be a neighborhood of $\nu \in M(T, X)$. Define

$$X_{n,F} := \{x \in X \mid \Upsilon_n(x) \in F\},$$

and define

$N(F; \delta, n, \varepsilon) :=$ maximal cardinality of a (δ, n, ε) -separated subset of $X_{n,F}$.

Definition 2.5. We say that the dynamical system T satisfies *uniform separation property*, if following holds. For any $\eta > 0$, there exist $\delta^* > 0, \epsilon^* > 0$ such that for μ ergodic and any neighborhood $F \subseteq M(X)$ of μ , there exists $n_{F,\mu,\eta}^*$, such that for $n \geq n_{F,\mu,\eta}^*$,

$$N(F; \delta^*, n, \epsilon^*) \geq 2^{n(h_\mu(f) - \eta)}.$$

Lemma 2.6. [65, Corollary 3.1] *Assume that (X, d) has the uniform separation property, and has entropy-dense property. For any η , there exist $\delta^* > 0, \epsilon^* > 0$ such that for $\mu \in M(T, X)$ and any neighborhood $F \subseteq M(X)$ of μ , there exists $n_{F,\mu,\eta}^*$, such that for $n \geq n_{F,\mu,\eta}^*$,*

$$N(F; \delta^*, n, \epsilon^*) \geq 2^{n(h_\mu(f) - \eta)}.$$

2.5. Recurrence. In the study of dynamical system and ergodic theory, recurrence plays important role, for example, Poincaré recurrent theorem. Given $x \in X$, let $\omega_T(x)$ denote the ω -limit set.

Definition 2.7. We call $x \in X$ to be *recurrent*, if $x \in \omega_T(x)$. A point $x \in X$ is called *wandering*, if there is a neighborhood U of x such that the sets $T^{-n}U$, $n \geq 0$, are mutually disjoint. Otherwise, x is called *non-wandering*.

Let Rec denote the set of all recurrent points, called *recurrent set*. Let Ω denote the set of all non-wandering points, called *non-wandering set*.

Now we start to introduce various kind of recurrence.

Definition 2.8. A point $x \in X$ is called *transitive*, if $\omega_T(x) = X$. A point is called *periodic* if it returns itself through finite iterates. A point $x \in X$ is *almost periodic*, if for every open neighborhood U of x , there exists $N > 0$ such that $f^k(x) \in U$ for some $k \in [n, n + N]$ and every integer $n \geq 1$.

Let $Tran$ denote the set of all transitive points. It is in general either empty or residual in the whole space. Let Per denote the set of periodic points and let A denote the set of all almost periodic points. In general periodic point maybe not exist but almost periodic point always exists. It is well-known from [11] that a point x is almost periodic if and only if x is minimal, i.e., $x \in \omega_T(x)$ and $\omega_T(x)$ is minimal (see [40, 38, 39, 55] for more related discussion in the sense that the space X is more general, not necessarily being compact metric space). Here an invariant set $E \subseteq X$ is called *minimal*, if for every point $y \in E$, $\omega_T(y) = E$. In particular, if X is minimal, we say the system T to be minimal. Remark that the almost periodic set A can be written as the union of all minimal sets.

If $E \subseteq X$ is nonempty and $x \in X$, define

$$\underline{P}_x(E) := \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x)) \quad \text{and} \quad \overline{P}_x(E) := \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(T^i(x)).$$

If $\underline{P}_x(E) = \overline{P}_x(E)$, we denote by $P_x(E)$, which means the probability of the orbit of x enters in E . Let $V_\varepsilon(x)$ denote ε -neighborhood of x , i.e., $V_\varepsilon(x) = \{y \in M \mid d(x, y) < \varepsilon\}$.

Definition 2.9. ((quasi-)weakly almost periodic) We call x to be a weakly almost periodic point, if for any $\varepsilon > 0$,

$$\underline{P}_x(V_\varepsilon(x)) > 0.$$

x is called to be a quasi-weakly almost periodic point, if for any $\varepsilon > 0$,

$$\overline{P}_x(V_\varepsilon(x)) > 0.$$

Let QW and W denote the sets of all weakly almost periodic points and all quasi-weakly almost periodic points, respectively. Now we introduce some equivalent definitions. Let $S \subseteq \mathbb{N}$, define

$$\bar{d}(S) := \limsup_{n \rightarrow \infty} \frac{\#(S \cap \{0, 1, \dots, n-1\})}{n}, \quad \underline{d}(S) := \liminf_{n \rightarrow \infty} \frac{\#(S \cap \{0, 1, \dots, n-1\})}{n},$$

where $\#Y$ denotes the cardinality of the set Y . These two concepts are called *upper density* and *lower density* of S , respectively. If $\bar{d}(S) = \underline{d}(S) = d$, we call S to have density of d . Define

$$B^*(S) := \limsup_{\#I \rightarrow \infty} \frac{\#(S \cap I)}{\#I}, \quad B_*(S) := \liminf_{\#I \rightarrow \infty} \frac{\#(S \cap I)}{\#I},$$

where $I \subseteq \mathbb{N}$ denotes finite continuous integer interval. These two concepts are called *Banach upper density* and *Banach lower density* of S , respectively. A set $S \subseteq \mathbb{N}$ is called *syndetic*, if there is $N > 0$ such that for any $n \geq 1$,

$$S \cap \{n, n+1, \dots, n+N\} \neq \emptyset.$$

Let $U, V \subseteq X$ be two nonempty open subsets and $x \in X$. Define sets of recurrent time

$$N(U, V) := \{n \geq 1 \mid U \cap f^{-n}(V) \neq \emptyset\} \quad \text{and} \quad N(x, U) := \{n \geq 1 \mid f^n(x) \in U\}.$$

Then it is easy to check that for any $x \in X$,

$$\begin{aligned} x \in A &\Leftrightarrow \forall \epsilon > 0, N(x, V_\epsilon(x)) \text{ is syndetic} \Leftrightarrow \forall \epsilon > 0, B_*(N(x, V_\epsilon(x))) > 0, \\ x \in W &\Leftrightarrow \forall \epsilon > 0, N(x, V_\epsilon(x)) \text{ has positive lower density,} \\ x \in QW &\Leftrightarrow \forall \epsilon > 0, N(x, V_\epsilon(x)) \text{ has positive upper density,} \\ x \in Tran &\Leftrightarrow \forall \text{ nonempty open set } U \subseteq X, N(x, U) \neq \emptyset, \\ x \in Rec &\Leftrightarrow \forall \epsilon > 0, N(x, V_\epsilon(x)) \neq \emptyset, \\ x \in \Omega &\Leftrightarrow \forall \epsilon > 0, N(V_\epsilon(x), V_\epsilon(x)) \neq \emptyset. \end{aligned}$$

We also introduce another concept called Banach recurrent. A point x is called *Banach recurrent*, if $\epsilon > 0$, $B^*(N(x, V_\epsilon(x))) > 0$. Let BR denote the set of all Banach recurrent points. Remark that

$$(2.7) \quad \Omega \supseteq Rec \supseteq BR \supseteq QW \supseteq W \supseteq A \supseteq Per \quad \text{and} \quad \Omega \supseteq Rec \supseteq Tran.$$

2.6. Regularity, Quasiregularity and Irregularity. Let us recall the concepts of *quasiregular points* (for example, see [25, 60]), *regular points* (see [60]) and *irregular points* (for example, see Chapter 8 in [3]).

Firstly let us recall the definition of generic point and quasiregular point (see [25, Chapter 4]).

Definition 2.10. A point $x \in X$ is said to be *generic* for a measure $\mu \in M(T, X)$, if for any continuous function $\phi : X \rightarrow \mathbb{R}$, the limit

$$\phi^*(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x))$$

exists and equals to $\int \phi d\mu$.

Let G_μ (or $G(\mu)$) denote the set of all generic points for μ , called generic set of μ for convenience. By weak* topology,

$$(2.8) \quad x \in G_\mu \Leftrightarrow \lim_{n \rightarrow \infty} \Upsilon_n(x) = \mu \Leftrightarrow M_x = \{\mu\}.$$

Recall a classical result (see [15, Theorem 3]) that every ergodic measure $\mu \in M_{erg}(T, X)$ satisfies that

$$(2.9) \quad h_\mu(f) = h_\mu(G_\mu).$$

Definition 2.11. A point $x \in X$ is called *quasiregular* with respect to T , if it is generic with respect to some invariant measure. Otherwise, x is called *irregular*.

Denote by QR the set of all quasiregular points with respect to T , called quasiregular set for convenience. By definition,

$$(2.10) \quad x \in QR \Leftrightarrow \exists \mu \in M(T, X), x \in G_\mu \Leftrightarrow M_x \text{ is a singleton.}$$

For convenience, for any $x \in QR$, denote by μ_x the invariant measure for which x is generic. Let I denote the complementary set of quasiregular set, that is $I = X \setminus QR$. This set is called *irregular set* and is composed of irregular points (for example, see [3, Chapter 8], also called *point with historic behavior* in [70, 77] and called ‘non-typical’ point in [7]). By (2.10) we have

$$(2.11) \quad x \in I \Leftrightarrow M_x \text{ is not a singleton} \Leftrightarrow \Upsilon_n(x) \text{ does not converge.}$$

Now we start to recall the concept of density point and regular point (see [60]). Let $QR_{erg} := \bigcup_{\mu \in M_{erg}(T, X)} G_\mu$. In [60] the point in QR_{erg} is called transitive, but in present paper transitive point means that its orbit is dense in the whole space X . To avoid confusion, in this paper points in QR_{erg} are called *ergodic-transitive* and the set QR_{erg} is called ergodic-transitive set, same name as in [82].

Definition 2.12. A point $x \in QR$ is called a *point of density*, if $\mu_x(U) > 0$ for every open set $U \subseteq X$ containing x .

Let QR_d denote the set of all points of density in QR and for convenience in present paper QR_d is called density set. It is easy to check that for any $x \in QR$,

$$(2.12) \quad x \in QR_d \Leftrightarrow x \in S_{\mu_x},$$

$$(2.13) \quad QR_d = \bigcup_{\mu \in M(T, X)} (G_\mu \cap S_\mu).$$

Definition 2.13. A point $x \in X$ is called *regular*, if it belongs to the set $R := QR_d \cap QR_{erg}$ (called regular set).

Remark that

$$(2.14) \quad R = \bigcup_{\mu \in M_{erg}(T, X)} (G_\mu \cap S_\mu) \subseteq QR_d \cup QR_{erg} \subseteq QR.$$

2.7. Lyapunov-level set and Lyapunov-irregular set w.r.t. asymptotically additive functions.

Lemma 2.14. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. Then for any invariant μ , the limit $\chi_\Phi(\mu) := \lim_{n \rightarrow \infty} \frac{1}{n} \int \phi_n(x) d\mu$ exists and the function $\chi_\Phi(\cdot) : M(T, X) \rightarrow \mathbb{R}$ is continuous.

Proof. The proof is standard from the definition of asymptotically additive, we refer [32] for the details. \square

Let \mathcal{AA} denotes the space of all asymptotically additive functions.

Theorem 2.15.

$$(2.15) \quad QR = \bigcup_{\mu \in M(T, X)} G_\mu = \bigcap_{\Phi \in \mathcal{AA}} R_\Phi,$$

$$(2.16) \quad I = X \setminus QR = \bigcap_{\mu \in M(T, X)} (X \setminus G_\mu) = \bigcup_{\Phi \in \mathcal{AA}(X)} X \setminus R_\Phi = \bigcup_{\Phi \in \mathcal{AA}} I_\Phi.$$

Proof. The two arguments are equivalent. So we just need to consider the first one. It can be deduced from the weak* topology and the definition of asymptotically additive. \square

Some classical results are known on multi-fractal analysis, for example see [65, 78, 59] for ergodic averages of additive functions and see [4, 32] for the case of asymptotically additive functions. Recall Lyapunov-level set

$$R_{\Phi,a} := \{x \in X \mid \lim_{n \rightarrow \infty} \frac{1}{n} \phi_n(x) = a\}.$$

Let us state some basic facts as follows.

Lemma 2.16. *Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. Then*

(1) *For $a \in \mathbb{R}$,*

$$(2.17) \quad x \in R_{\Phi,a} \Leftrightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \phi_n(x) = a \Leftrightarrow M_x \subseteq \{\rho \mid \chi_{\Phi}(\rho) = a\}.$$

(2)

$$(2.18) \quad x \in R_{\Phi} \Leftrightarrow \exists a \in \mathbb{R}, \quad M_x \subseteq \{\rho \mid \chi_{\Phi}(\rho) = a\}.$$

(3)

$$(2.19) \quad x \in I_{\Phi} \Leftrightarrow \exists \mu_1, \mu_2 \in M_x \text{ such that } \chi_{\Phi}(\mu_1) \neq \chi_{\Phi}(\mu_2).$$

(4)

$$(2.20) \quad I_{\Phi} \neq \emptyset \Rightarrow \inf_{\mu \in M(T,X)} \chi_{\Phi}(\mu) < \sup_{\mu \in M(T,X)} \chi_{\Phi}(\mu).$$

(5) *If $I_{\Phi} \neq \emptyset$, $\text{Int}(I_{\Phi})$ is a nonempty and open interval.*

Proof. We only need to prove (1), since (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5). (1) can be deduced from the weak* topology and the definition of asymptotically additive. \square

Lemma 2.17. *Let K be a compact connected subset of $M(T, X)$ and let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions.*

(1) *If $\inf_{\omega \in K} \chi_{\Phi}(\omega) < \sup_{\omega \in K} \chi_{\Phi}(\omega)$, then $G_K \subseteq I_{\Phi}$.*

(2) *If $\inf_{\omega \in K} \chi_{\Phi}(\omega) = \sup_{\omega \in K} \chi_{\Phi}(\omega) = a$ for some $a \in \mathbb{R}$, then $G_K \subseteq R_{\Phi,a}$.*

Proof. (1) is from (2.19) and (2) is from (2.17). \square

Recall that the system T has *single-saturated* property or T is *single-saturated*, if for any $\mu \in M(T, X)$,

$$(2.21) \quad h_{\text{top}}(T, G_{\mu}) = h_{\mu}.$$

Now let us state a result on variational principle of $R_{\Phi,a}$. Let

$$H_{\Phi,a} = \sup\{h_{\rho} \mid \rho \in M(T, X) \text{ and } \chi_{\Phi}(\rho) = a\}.$$

Proposition 2.18. *(Variational Principle on level sets of Lyapunov exponents) Suppose that T is single-saturated. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. Then for any real number $a \in L_{\Phi}$,*

$$h_{\text{top}}(T, R_{\Phi,a}) = H_{\Phi,a}.$$

Proof. For any real number $t \geq 0$, define the (maybe empty) set

$$Q(t) := \{x : \exists \mu \in M_x \text{ s.t. } h_\mu(f) \leq t\}.$$

From [15, Theorem 2]: $h_{top}(f, Q(t)) \leq t$. Let $t = H_{\Phi, a}$, notice that from (2.17) $R_{\Phi, a} \subseteq Q(t)$ and thus $h_{top}(T, R_{\Phi, a}) \leq h_{top}(T, Q(t)) \leq t$.

On the other hand, for any invariant measure ρ with $\chi_\Phi(\rho) = a$, by (2.17) we get that $G_\rho \subseteq R_{\Phi, a}$. By assumption of single-saturated, $h_{top}(T, R_{\Phi, a}) \geq h_{top}(T, G_\rho) = h_\rho$. \square

Let us consider the entropy map $\Psi : L_\Phi \rightarrow \mathbb{R} : a \mapsto H_{\Phi, a}$.

Proposition 2.19. *Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions with $I_\Phi(T) \neq \emptyset$. Then the entropy map $\Psi : L_\Phi \rightarrow \mathbb{R}$ is a concave function and thus continuous on $\text{Int}(L_\Phi)$. In particular, $h_{top}(T) = \sup_{a \in \text{Int}(L_\Phi)} \Psi(a)$.*

Proof. The proof is similar as the one of [82, Proposition 4.9], just replacing the integral of a function there by $\chi_\Phi(\mu)$. For more details, see [82]. \square

From Birkhoff ergodic theorem, QR has totally full measure. Thus by (2.16) I has zero measure for any invariant measure and so does every I_Φ . However, in recent several years many people have focused on studying the dynamical complexity of irregular set from different sights, for example, in the sense of dimension theory and topological entropy (or pressure) etc. Pesin and Pitskel [63] are the first to notice the phenomenon of the irregular set carrying full topological entropy in the case of the full shift on two symbols. Barreira, Schmeling, etc. studied the irregular set in the setting of subshifts of finite type and beyond, see [7, 3, 78] etc. Recently, Thompson shows in [80, 81] that every Φ -irregular set I_Φ (for the case of additive functions) either is empty or carries full topological entropy (or pressure) when the system satisfies (almost) specification, which is inspired from [65] by Pfister and Sullivan and [78] by Takens and Verbitskiy. Now we state a result for the asymptotically additive case.

Lemma 2.20. *Suppose that T is saturated. Let $\Psi := \{\psi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. If*

$$\inf_{\mu \in M(T, X)} \chi_\Psi(\mu) < \sup_{\mu \in M(T, X)} \chi_\Psi(\mu),$$

then $h_{top}(T, I_\Psi) = h_{top}$. In particular,

$$\inf_{\mu \in M(T, X)} \chi_\Psi(\mu) < \sup_{\mu \in M(T, X)} \chi_\Psi(\mu) \Leftrightarrow I_\Psi \neq \emptyset.$$

Proof. Take two invariant measures ν_1, ν_2 such that $\chi_\Psi(\nu_1) \neq \chi_\Psi(\nu_2)$. Fix $\epsilon > 0$. Take an invariant measure μ with positive entropy larger than $h_{top} - \epsilon$ and choose $\theta \in (0, 1)$ close to 1 such that $\theta h_\mu > h_{top} - \epsilon$. Define measures $\mu_i := \frac{1}{2}(\mu + \nu_i)$ ($i = 1, 2$), then $\chi_\Psi(\mu_1) \neq \chi_\Psi(\mu_2)$ and $\min\{h_{\mu_1}, h_{\mu_2}\} \geq \theta h_\mu > h_{top} - \epsilon$. Let $K = \{t\mu_1 + (1-t)\mu_2 \mid t \in [0, 1]\}$. Since T is saturated, one can get $h_{top}(T, I_\Psi) \geq h_{top}(T, G_K) = \min_{i=1,2} h_{\mu_i}(T) > h_{top} - \epsilon$.

For the case of ' \Leftarrow ', it is the fact (2.20). For the case of ' \Rightarrow ', it can be deduced from $h_{top}(T, I_\Psi) = h_{top} > 0$, since in this paper we assume T to have positive entropy. \square

3. SOME BASIC FACTS

3.1. **Various saturated property.** For convenience, we use

$$S, LS, TS, LTS$$

to denote saturated, locally-saturated, transitively-saturated, locally-transitively-saturated;

$$CS, LCS, TCS, LTCS$$

to denote convex-saturated, locally-convex-saturated, transitively-convex-saturated, locally-transitively-convex-saturated; and

$$SS, LSS, TSS, LTSS$$

to denote single-saturated, locally-single-saturated, transitively-single-saturated, locally-transitively-single-saturated, respectively.

Lemma 3.1. *Let $T : X \rightarrow X$ be a continuous map of a compact metric space X . Then*

$$\begin{array}{ccccccc} TS & \Leftrightarrow & LTS & \Rightarrow & LS & \Rightarrow & S \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ TCS & \Leftrightarrow & LTCS & \Rightarrow & LCS & \Rightarrow & CS \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ TSS & \Leftrightarrow & LTSS & \Rightarrow & LSS & \Rightarrow & SS \end{array}$$

Proof. Given a compact and connected subset $K \subseteq M(X, T)$ we only need to prove that

(1) $h_{top}(T, G_K^T) = \inf\{h_\mu(T) \mid \mu \in K\} \Leftrightarrow h_{top}(T, G_K^T \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$.

(2) $h_{top}(T, G_K^T \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X \Rightarrow h_{top}(T, G_K \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X \Rightarrow h_{top}(T, G_K) = \inf\{h_\mu(T) \mid \mu \in K\}$.

For part (1), on one hand, if $h_{top}(T, G_K^T \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$, then taking $U = X$, we have $h_{top}(T, G_K^T) = h_{top}(T, G_K^T \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$. On the other hand, if $h_{top}(T, G_K^T) = \inf\{h_\mu(T) \mid \mu \in K\}$, then $h_{top}(T, G_K^T \cap U) \leq h_{top}(T, G_K^T) = \inf\{h_\mu(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$. Now we are going to prove that for any nonempty open set $U \subseteq X$, $h_{top}(T, G_K^T \cap U) \geq h_{top}(T, G_K^T)$. Notice that by (2.2) and (2.3) for any $n \geq 1$,

$$h_{top}(T, f^{-n}(U \cap G_K^T)) = h_{top}(T, f^n f^{-n}(U \cap G_K^T)) \leq h_{top}(T, U \cap G_K^T)$$

and by the definition of transitivity and invariance of G_K^T (i.e., $f^{\pm 1}G_K^T \subseteq G_K^T$),

$$G_K^T = G_K^T \cap \left(\bigcup_{n \geq 0} f^{-n}U \right) = \bigcup_{n \geq 0} f^{-n}(U \cap G_K^T).$$

Thus by (2.4)

$$\begin{aligned} h_{top}(T, G_K^T) &= h_{top}(T, \bigcup_{n \geq 0} f^{-n}(U \cap G_K^T)) \\ &= \sup_{n \geq 0} h_{top}(T, f^{-n}(U \cap G_K^T)) \leq h_{top}(T, U \cap G_K^T). \end{aligned}$$

For part (2), on one hand, if $h_{top}(T, G_K^T \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$, then $h_{top}(T, G_K \cap U) \geq h_{top}(T, G_K^T \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$. Recall that in general $h_{top}(f, G_K) \leq \inf\{h_\mu(f) \mid \mu \in K\}$ (see [64, Theorem 4.1 (3)]), thus $h_{top}(T, G_K \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$. On the other hand, if $h_{top}(T, G_K \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$, then taking $U = X$, we have $h_{top}(T, G_K) = h_{top}(T, G_K^T \cap U) = \inf\{h_\mu(T) \mid \mu \in K\}$. \square

Remark 3.2. If $G_K^T \neq \emptyset$, then there is transitive point so that the system is transitive and thus is surjective. So for $n \geq 1$,

$$h_{top}(T, f^{-n}(U \cap G_K^T)) = h_{top}(T, f^n f^{-n}(U \cap G_K^T)) = h_{top}(T, U \cap G_K^T).$$

By definition and the system being surjective, one also see that $f^{\pm 1}G_K = G_K$, $f^{\pm 1}Tran = Tran$, $f^{\pm 1}G_K^T = G_K^T$.

3.2. Asymptotically additive functions. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions with $Int(L_\Phi) \neq \emptyset$ and $\mathcal{R} \subseteq M(T, X)$, we say that \mathcal{R} has *Property (P)* w.r.t. Φ , if for every $\nu \in M(T, X)$, there is $\rho \in \mathcal{R}$ such that $\chi_\Phi(\rho) \neq \chi_\Phi(\nu)$, and if moreover, for any $n \in \mathbb{Z}^+$ and $a \in Int(L_\Phi)$, there is $\mu_1, \mu_2, \dots, \mu_n \in \mathcal{R}$ and $\nu_1, \nu_2, \dots, \nu_n \in \mathcal{R}$ such that

$$\chi_\Phi(\mu_n) < \dots < \chi_\Phi(\mu_1) < a < \chi_\Phi(\nu_1) < \dots < \chi_\Phi(\nu_n).$$

By (2.20) $I_\Phi(T) \neq \emptyset \Rightarrow Int(L_\Phi) \neq \emptyset$. Since $\chi_\Phi(\nu)$ is continuous w.r.t. ν (by Lemma 3.3), it is easy to see that if \mathcal{R} is a dense subset of $M(T, X)$, then $\{\chi_\Phi(\nu) \mid \nu \in \mathcal{R}\}$ is dense in L_Φ . This implies that

Lemma 3.3. *Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions with $I_\Phi(T) \neq \emptyset$ and let \mathcal{R} be a dense subset of $M(T, X)$. Then \mathcal{R} has Property (P) w.r.t. Φ .*

3.3. Characterization of recurrence. A point x is called *quasi-generic* for some measure μ , if there are two sequences of positive integers $\{a_k\}$ and $\{b_k\}$ with $b_k \geq a_k$, $\lim_{k \rightarrow \infty} b_k - a_k = \infty$ such that

$$\lim_{k \rightarrow \infty} \frac{1}{b_k - a_k + 1} \sum_{j=a_k}^{b_k} \delta_{T^j(x)} = \mu$$

in weak* topology. Let $M_x^* = \{\mu \in M(T, X) : x \text{ is quasi-generic for } \mu\}$. This concept is from [36] and from there it is known M_x^* is always nonempty, compact and connected. Note that $M_x \subseteq M_x^*$. Let $C_x = \bigcup_{m \in M_x} S_m$ and $C_x^* = \bigcup_{m \in M_x^*} S_m$. From [88, 87, 89, 90, 46] we know some basic facts:

Lemma 3.4.

$$(3.1) \quad \forall x \in X, \quad C_x \subseteq C_x^* \subseteq \omega_T(x);$$

$$(3.2) \quad x \in W \Leftrightarrow x \in C_x = S_\mu, \forall \mu \in M_x \Leftrightarrow x \in \omega_T(x) = S_\mu, \forall \mu \in M_x;$$

$$(3.3) \quad x \in QW \Leftrightarrow x \in C_x \Leftrightarrow x \in \omega_T(x) = C_x$$

$$(3.4) \quad x \in A \Leftrightarrow x \in C_x^* = S_\mu, \forall \mu \in M_x^* \Leftrightarrow x \in \omega_T(x) = S_\mu, \forall \mu \in M_x^*;$$

$$(3.5) \quad x \in BR \Leftrightarrow x \in C_x^* \Leftrightarrow x \in \omega_T(x) = C_x^*$$

Corollary 3.5. *For any $x \in X$,*

$$(3.6) \quad C_x = X \Rightarrow x \in QW \cap Tran.$$

Proof. If $C_x = X$, by (3.1) $\omega_T(x) = X$ and $x \in X = C_x$ so that by (3.3), $x \in QW \cap Tran$.

If $C_x^* = X$, by (3.1) $\omega_T(x) = X$ and $x \in X = C_x^*$ so that by (3.5), $x \in BR \cap Tran$. \square

Let $BR^* := \{x \in BR \mid \exists \mu \in M_x^* \text{ s.t. } S_\mu = C_x^*\}$.

Lemma 3.6. *If ergodic measures are dense in the space of invariant measures (or T has entropy-dense property), then for any $x \in \text{Tran}$, $M_x^* = M(T, X)$. If further T has an invariant measure with full support, then $\text{Tran} \subseteq BR^*$ (or $\text{Tran} \subseteq BR^* \setminus A$).*

Proof. From [36, Proposition 3.9, Page 65] we know that for a point x_0 and an ergodic measure $\mu_0 \in M(\omega_T(x_0), T)$, x_0 is quasi-generic for μ_0 . This implies that for any $x \in \text{Tran}$, $M_{\text{erg}}(T, X) \subseteq M_x^*$. By assumption of density of ergodic measures, $M_x^* = M(T, X)$. If further T has an invariant measure with full support, then $C_x^* = X = \omega_T(x)$ and by (3.5) $x \in BR^*$. \square

3.4. Characterization of different G_K . For a set $K \subseteq M(T, X)$, define $C_K = \overline{\bigcup_{\omega \in K} S_\omega}$. Recall the notions that $G_K = \{x \in X \mid M_x = K\}$, $G_K^T = \{x \in \text{Tran} \mid M_x = K\}$.

Lemma 3.7. *Let K be a compact connected subset of $M(T, X)$.*

- (1) *If for any $\omega \in K$, $S_\omega = X$, then $G_K^T = G_K \subseteq W \cap \text{Tran}$.*
- (2) *If there are two measures $\omega_i \in K$ ($i = 1, 2$), $S_{\omega_1} \subsetneq S_{\omega_2} = X$, then $G_K^T = G_K \subseteq (V \setminus W) \cap \text{Tran}$.*
- (3) *If $C_K = X$ but for any $\omega \in K$, $S_\omega \neq X$, then $G_K^T = G_K \subseteq (QW \setminus V) \cap \text{Tran}$.*
- (4) *If $C_K \neq X$ and $T|_{C_K}$ is not a transitive subsystem, then $G_K \subseteq X \setminus QW$. It follows $G_K^T \subseteq \text{Tran} \setminus QW$.*
- (5) *If $C_K \neq X$, then $G_K^T \subseteq \text{Tran} \setminus QW$.*

Proof. The proofs are not difficult and readers can use Lemma 3.4 to give the proofs. For example, we give the proof of (4).

For case (4), by contradiction there is $x \in G_K \cap QW$, then $C_x = C_K$ and by (3.3) $x \in \omega_T(x) = C_x$ so that $x \in \omega_T(x) = C_K$. It means $T|_{C_K}$ is transitive, it contradicts the assumption. \square

3.5. Full support and minimal set.

Lemma 3.8. *Suppose that T is single-saturated and there is some invariant measure μ with full support (i.e., $S_\mu = X$). Then T is transitive.*

Proof. If μ is ergodic, transitivity is obvious. However, μ may be not ergodic. Since in this paper we always assume T has positive entropy, we can take invariant ν s.t. $h_\nu > 0$. Take $\omega = \frac{1}{2}(\mu + \nu)$, then $h_\omega > 0$ and $S_\omega = X$. By assumption, $h_\omega = h_{\text{top}}(T, G_\omega)$. It follows that $G_\omega \neq \emptyset$. Take $x \in G_\omega$, then $X = S_\omega \subseteq \omega_T(x)$ so that T is transitive. \square

Lemma 3.9. *Suppose that a subset B' of $B := \{\omega \in M(T, X) \mid S_\omega \neq X\}$ is dense in $M(T, X)$. Then there is some invariant measure μ with full support (i.e., $S_\mu = X$) $\Leftrightarrow \overline{\bigcup_{\omega \in B'} S_\omega} = X$.*

Proof. \Rightarrow By assumption there is a sequence of invariant measures $\mu_i \in B'$ with $S_{\mu_i} \neq X$ converging to μ . Then $1 = \limsup_{n \rightarrow \infty} \mu_n(\overline{\bigcup_{\omega \in B'} S_\omega}) \leq \mu(\overline{\bigcup_{\omega \in B'} S_\omega})$. It follows that $X = S_\mu \subseteq \overline{\bigcup_{\omega \in B'} S_\omega}$.

\Leftarrow Take a sequence of invariant measures $\mu_i \in B'$ with $S_{\mu_i} \neq X$ such that $\overline{\bigcup_{i \geq 1} S_{\mu_i}} = X$. Let $\mu = \sum_{n \geq 1} \frac{1}{2^n} \mu_n$. Then $\mu(\bigcup_{i \geq 1} S_{\mu_i}) = 1$ so that $S_\mu = X$. \square

Lemma 3.10. *Suppose that T has entropy-dense property and there is some invariant measure μ with full support (i.e., $S_\mu = X$). Then $B := \{\omega \in M_{\text{erg}}(T, X) \mid S_\omega \neq X\}$ is dense in $M(T, X)$ and $\overline{\bigcup_{\omega \in B} S_\omega} = X$.*

Proof. By entropy-dense property, the set B is dense in $M(T, X)$. Then by Lemma 3.9 $\overline{\bigcup_{\omega \in B} S_\omega} = X$. \square

Lemma 3.11. *Suppose that $\{\mu \in M(T, X) \mid S_\mu \text{ is minimal}\}$ is dense in $M(T, X)$. Then there is some invariant measure μ with full support (i.e., $S_\mu = X$) \Leftrightarrow almost periodic set A is dense in X .*

Proof. \Rightarrow By assumption there is a sequence of invariant measures μ_i with $S_{\mu_i} \subseteq A$ converging to μ . Then $1 = \limsup_{n \rightarrow \infty} \mu_n(A) \leq \limsup_{n \rightarrow \infty} \mu_n(\overline{A}) \leq \mu(\overline{A})$. It follows that $X = S_\mu \subseteq \overline{A}$.

\Leftarrow Take a sequence of points $\{x_i\} \subseteq A$ dense in X . For any i , take μ_i to be a invariant measure on $\omega_T(x_i)$. Then $x_i \in \omega_T(x_i) = S_{\mu_i}$ and so $\bigcup_{i \geq 1} S_{\mu_i} = X$. Let $\mu = \sum_{n \geq 1} \frac{1}{2^n} \mu_n$. Then $\mu(\bigcup_{i \geq 1} S_{\mu_i}) = 1$ so that $S_\mu = X$. \square

Lemma 3.12. *Suppose that T has g -almost product property. Then ergodic measures supported on minimal sets are dense in $M(T, X)$.*

Proof. Let $\nu \in M(T, X)$ and $G \subseteq M(X)$ be a neighborhood of ν . Take an open ball $G' \subseteq M(X)$ such that $\nu \in G' \subseteq \overline{G'} \subset G$. From the proof of [64, Proposition 2.3 (1)], one construct a closed invariant set Y and there exists $n_{G'} \in \mathbb{N}$ such that for any $y \in Y$ and any $n \geq n_{G'}$, $\Upsilon_n(y) \in G'$. So for any $m \in M_{erg}(T, Y)$, by Birkhoff ergodic theorem there is $y \in Y$ such that $\Upsilon_n(y)$ converge to m in weak* topology and thus $m \in \overline{G'}$. In other words, $M_{erg}(T, Y) \subseteq \overline{G'}$. By convex property of the ball G' and Ergodic Decomposition theorem, $M(T, Y) \subseteq \overline{G'}$. Take an ergodic measure μ supported on a minimal subset of Y , then $\mu \in \overline{G'} \subseteq G$. \square

Lemma 3.13. *Suppose that T has g -almost product property and there is some invariant measure μ with full support (i.e., $S_\mu = X$). Then ergodic measures supported on minimal sets are dense in $M(T, X)$ and almost periodic set A is dense in X .*

Proof. by Lemma 3.12 ergodic measures supported on minimal sets are dense in $M(T, X)$. Combining with Lemma 3.11, almost periodic set A is dense in X . \square

4. GAP-SETS W.R.T. I_Φ

In this section we discuss the entropy estimate on gap-sets w.r.t. I_Φ .

Proposition 4.1. *Suppose that T is convex-saturated and there is some invariant measure μ with full support (i.e., $S_\mu = X$). Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. Then either $I_\Phi = \emptyset$ or*

$$h_{top}(T, (I_\Phi \cap Tran \cap W) \setminus QR) = h_{top}(T, I_\Phi) = h_{top}.$$

Proof. Suppose $I_\Phi \neq \emptyset$ and fix $\varepsilon > 0$. By classical Variational Principle (Theorem 8.6 in [83]), we can take $\mu \in M(T, X)$ such that

$$h_\mu > h_{top} - \varepsilon.$$

By Lemma 3.3, we can take $\nu \in M(T, X)$ such that $S_\nu = X$ and $\chi_\Phi(\nu) \neq \chi_\Phi(\mu)$. Then we can choose two different numbers $0 < \theta_1 < \theta_2 < 1$ close to 1 enough such that for $\omega_i = \theta_i \mu + (1 - \theta_i) \nu$, $i = 1, 2$, one has

$$(4.1) \quad h_{\omega_i} = \theta_i h_\mu + (1 - \theta_i) h_\nu \geq \theta_i h_\mu > h_{top} - \varepsilon, \quad i = 1, 2.$$

Remark that $\theta_1 \neq \theta_2$ and $\chi_\Phi(\nu) \neq \chi_\Phi(\mu)$ imply

$$(4.2) \quad \chi_\Phi(\omega_1) \neq \chi_\Phi(\omega_2);$$

and $S_\nu = X$ implies

$$(4.3) \quad S_{\omega_i} = S_\mu \cup S_\nu = X, \quad i = 1, 2.$$

Let

$$K = \{\tau \omega_1 + (1 - \tau) \omega_2 \mid \tau \in [0, 1]\}.$$

Then by (4.3) and (4.1) for any $m = \tau\omega_1 + (1 - \tau)\omega_2 \in K$,

$$S_m = X, \quad h_m \geq \min\{h_{\omega_1}, h_{\omega_2}\} > h_{top} - \varepsilon.$$

By (4.2), Lemma 2.17 (1) and Lemma 3.7 (1) $G_K \subseteq I_\Phi \cap W \cap Tran$. Since T is convex-saturated, then

$$h_{top}(T, I_\Phi \cap Tran \cap W) \geq h_{top}(T, G_K) = \inf\{h_m \mid m \in K\} \geq h_{top} - \varepsilon.$$

Note that $(I_\Phi \cap Tran \cap W) \setminus QR = I_\Phi \cap Tran \cap W$, since $QR \cap I_\Phi = \emptyset$. We complete the proof. \square

Proposition 4.2. *Suppose that T is convex-saturated, has entropy-dense property and there is some invariant measure with full support. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. Then either $I_\Phi = \emptyset$ or*

$$h_{top}(T, I_\Phi \cap Tran \cap V \setminus W) = h_{top}(T, I_\Phi) = h_{top}.$$

Proof. Suppose $I_\Phi \neq \emptyset$ and fix $\varepsilon > 0$. By classical Variational Principle (Theorem 8.6 in [83]), we can take $\mu_0 \in M(T, X)$ such that

$$h_{\mu_0} > h_{top} - \varepsilon.$$

By entropy-dense property we can choose $\mu \in M_{erg}(T, X)$ (close to μ_0) such that $S_\mu \subsetneq X$ and

$$h_\mu > h_{top} - \varepsilon.$$

By Lemma 3.3, $I_\Phi \neq \emptyset$ implies that we can take $\nu \in M(T, X)$ such that $S_\nu = X$ and $\chi_\Phi(\nu) \neq \chi_\Phi(\mu)$. Then we can take $\theta \in (0, 1)$ close to 1 such that $h_\omega \geq \theta h_\mu > h_{top} - \varepsilon$ where $\omega = \theta\mu + (1 - \theta)\nu$. Remark that $S_\omega = S_\mu \cup S_\nu = X$ and

$$(4.4) \quad \chi_\Phi(\omega) \neq \chi_\Phi(\mu).$$

Let

$$K = \{\tau\omega + (1 - \tau)\mu \mid \tau \in [0, 1]\}.$$

Then for any $m = \tau\omega + (1 - \tau)\mu \in K$

$$h_m \geq \min\{h_\omega, h_\mu\} > h_{top} - \varepsilon.$$

Since $S_\mu \subsetneq S_\omega = X$ and $\mu, \omega \in K$, by Lemma 3.7 (2) $G_K \subseteq Tran \cap V \setminus W$. By (4.4) and Lemma 2.17 (1) $G_K \subseteq I_\Phi$. Since T is convex-saturated, then

$$h_{top}(T, I_\Phi \cap Tran \cap V \setminus W) \geq h_{top}(T, G_K) = \inf\{h_m \mid m \in K\} > h_{top} - \varepsilon.$$

\square

Proposition 4.3. *Suppose that T is saturated, has entropy-dense property and there is some invariant measure with full support. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. Then either $I_\Phi = \emptyset$ or*

$$h_{top}(T, Tran \cap I_\Phi \cap QW \setminus V) = h_{top}(T, I_\Phi) = h_{top}.$$

Compared with [82, Proposition 7.3], this proposition does not require density of periodic points and measures, just assuming a measure with full support instead.

Proof. Suppose $I_\Phi \neq \emptyset$ and fix $\varepsilon > 0$. By classical Variational Principle (Theorem 8.6 in [83]), we can take $\mu_0 \in M(T, X)$ such that

$$h_{\mu_0} > h_{top} - \varepsilon.$$

By entropy-dense property, we can choose $\mu \in M_{erg}(T, X)$ (close to μ_0) such that $S_\mu \subsetneq X$ and

$$h_\mu > h_{top} - \varepsilon.$$

From entropy-dense property, the set $B := \{\mu \in M_{erg}(T, X) \mid S_\mu \neq X\}$ is dense in $M(T, X)$. By Lemma 3.3, if let $\mathcal{R} = B$, then $I_\Phi \neq \emptyset$ implies that we can take one ergodic measure m_1 with $S_{m_1} \neq X$ such that $\chi_\Phi(m_1) \neq \chi_\Phi(\mu)$.

By Lemma 3.10, $\overline{\cup_{\omega \in B} S_\omega} = X$. Since X and $M(T, X)$ are compact metric spaces, we can take a sequence of measures $\{m_i\}_{i \geq 2}$ contained in B such that the set $\cup_{n \geq 2} S_{m_i}$ is dense in X .

Take a strictly increasing sequence of $\{\theta_i \mid \theta_i \in (0, 1)\}_{i=1}^\infty$ such that

$$\lim_{i \rightarrow +\infty} \theta_i = 1$$

and

$$h_{\nu_i} \geq \theta_i h_\mu > h_{top} - \varepsilon$$

where $\nu_i = \theta_i \mu + (1 - \theta_i) m_i$, $i = 1, 2, 3, \dots$. Remark that for any i , $S_{\nu_i} = S_\mu \cup S_{m_i}$ and note that

$$(4.5) \quad \chi_\Phi(\nu_1) \neq \chi_\Phi(\mu).$$

Now we consider

$$K = \bigcup_{i \geq 1} \{\tau \nu_i + (1 - \tau) \nu_{i+1} \mid \tau \in [0, 1]\} \cup \{\mu\}.$$

Remark that K is a nonempty connected compact subset of $M(T, X)$ because $\nu_i \rightarrow \mu$ in weak* topology. Note that $C_K = \overline{\cup_{\omega \in K} S_\omega} = \overline{\cup_i S_{\nu_i} \cup S_\mu} = X$. By Lemma 3.8 we know T is transitive. Then any finite compact invariant sets $Z_j \subsetneq X$, $j = 1, 2, \dots, k$, $\cup_{j=1}^k Z_j \neq X$, since $\cup_{j=1}^k Z_j \cap Tran = \emptyset$. Then for any $\omega \in K$, there is some i such that $S_\omega \subseteq S_\mu \cup S_{m_i} \cup S_{m_{i+1}} \neq X$. By Lemma 3.7 (3) $G_K \subseteq Tran \cap QW \setminus V$. By (4.5) and Lemma 2.17 (1) $G_K \subseteq I_\Phi$. Since T is saturated, then

$$\begin{aligned} h_{top}(T, Tran \cap I_\Phi \cap QW \setminus V) &\geq h_{top}(T, G_K) = \inf\{h_m \mid m \in K\} \\ &= \min\{\inf_{i \geq 1}\{h_{\nu_i}\}, h_\mu\} \geq h_{top} - \varepsilon. \end{aligned}$$

□

Proposition 4.4. *Suppose that T is convex-saturated system with entropy-dense property, and there is an invariant measure with full support. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions. Then*

(1) *If T is transitively-convex-saturated, then either $I_\Phi = \emptyset$ or*

$$h_{top}(T, Tran \cap BR^* \cap I_\Phi \cap I \setminus QW) = h_{top}(T, I_\Phi) = h_{top}.$$

(2) *If $\{\mu \in M(T, X) \mid S_\mu \text{ is minimal}\}$ is dense in $M(T, X)$, then either $I_\Phi = \emptyset$ or*

$$h_{top}(T, I_\Phi \cap I \setminus QW) = h_{top}(T, I_\Phi) = h_{top}.$$

Compared with [82, Proposition 7.4], the second item of this proposition does not require density of periodic points and measures, just assuming a measure with full support instead. Moreover, the first item gives a new result for transitively-saturated case.

Proof. Suppose $I_\Phi \neq \emptyset$ and fix $\varepsilon > 0$. By classical Variational Principle (Theorem 8.6 in [83]), we can take $\mu_0 \in M(T, X)$ such that

$$h_{\mu_0} > h_{top} - \varepsilon.$$

By entropy-dense property, we can choose $\mu \in M_{erg}(T, X)$ (close to μ_0) such that $S_\mu \subsetneq X$ and

$$h_\mu > h_{top} - \varepsilon.$$

(1) By entropy-dense property, $M_{erg}(T, X)$ and $B := \{\omega \in M_{erg}(T, X) \mid S_\omega \neq X\}$ are dense subsets of $M(T, X)$. By Lemma 3.6, $Tran \subseteq BR$. By Lemma 3.3, if let $\mathcal{R} = B$, then $I_\Phi \neq \emptyset$ implies that we can take one ergodic measure m_1

with $S_{m_1} \neq X$ such that $\chi_\Phi(m_1) \neq \chi_\Phi(\mu)$. Take $\theta \in (0, 1)$ close to 1 such that $\omega = \theta\mu + (1 - \theta)m_1$ satisfies $h_\omega \geq \theta h_\mu > h_{top} - \varepsilon$. Then ω also satisfies that

$$(4.6) \quad \chi_\Phi(\omega) \neq \chi_\Phi(\mu).$$

By Lemma 3.8 we know T is transitive. Then any finite compact invariant sets $Z_j \subsetneq X$, $j = 1, 2, \dots, k$, $\cup_{j=1}^k Z_j \neq X$, since $\cup_{j=1}^k Z_j \cap Tran = \emptyset$. Then $S_\mu \cup S_\omega = S_\mu \cup S_{m_1} \neq X$.

Let $K = \{\tau\mu + (1 - \tau)\omega \mid \tau \in [0, 1]\}$. Note that $C_K = \overline{\cup_{\tau \in K} S_\tau} \subseteq S_\mu \cup S_{m_1} \neq X$. By Lemma 3.7 (5) $G_K^T \subseteq Tran \setminus QW$. By (4.6) and Lemma 2.17 (1) $G_K^T \subseteq I_\Phi$. Note that $I_\Phi \cap I = I_\Phi$. Since T is transitively-convex-saturated, then

$$\begin{aligned} h_{top}(T, Tran \cap BR^* \cap I_\Phi \cap I \setminus QW) &= h_{top}(T, Tran \cap I_\Phi \setminus QW) \\ &\geq h_{top}(T, G_K^T) = \inf_{m \in K} h_m \geq \min\{h_\mu, h_\omega\} > h_{top} - \varepsilon. \end{aligned}$$

(2) By Lemma 3.11, $\overline{A} = X$. Since $X \setminus S_\mu$ is nonempty and open, by density of almost periodic points

$$J := \{\nu \in M(T, X) \mid S_\nu \text{ is minimal, } S_\nu \setminus S_\mu \neq \emptyset\} \neq \emptyset.$$

Now we will construct an invariant measure κ such that the set $S_\kappa \setminus S_\mu$ is composed of one minimal set and

$$\chi_\Phi(\kappa) \neq \chi_\Phi(\mu).$$

More precisely, if there is a measure $\nu \in J$ such that $\chi_\Phi(\nu) \neq \chi_\Phi(\mu)$, then take $\kappa = \nu$ and remark that $S_\kappa \setminus S_\mu = S_\nu$. Otherwise, for any $\nu \in J$, $\chi_\Phi(\nu) = \chi_\Phi(\mu)$. Take such a measure ν . By assumption and Lemma 3.3, $I_\Phi \neq \emptyset$ imply

$$Y := \{\tau \mid \chi_\Phi(\tau) \neq \chi_\Phi(\mu), S_\tau \text{ is minimal}\} \neq \emptyset.$$

Then we can take $\nu' \in Y$ such that $\chi_\Phi(\nu') \neq \chi_\Phi(\mu)$. Remark that in this case $Y \cap J = \emptyset$ so that $S_{\nu'} \setminus S_\mu = \emptyset$. Then $S_{\nu'} \subseteq S_\mu$. So if we take $\kappa = \frac{1}{2}(\nu + \nu')$, then $\chi_\Phi(\kappa) \neq \chi_\Phi(\mu)$. Note that $S_\kappa = S_\nu \cup S_{\nu'}$ and $S_\kappa \setminus S_\mu = S_\nu$.

Take $\theta \in (0, 1)$ close to 1 such that $\omega = \theta\mu + (1 - \theta)\kappa$ satisfies $h_\omega \geq \theta h_\mu > h_{top} - \varepsilon$. Then ω also satisfies that

$$(4.7) \quad \chi_\Phi(\omega) \neq \chi_\Phi(\mu).$$

Remark that $S_\omega = S_\mu \cup S_\nu \cup S_{\nu'} = S_\mu \sqcup S_\nu$. Let $K = \{\tau\mu + (1 - \tau)\omega \mid \tau \in [0, 1]\}$. By Lemma 3.8 we know T is transitive. Then any finite compact invariant sets $Z_j \subsetneq X$, $j = 1, 2, \dots, k$, $\cup_{j=1}^k Z_j \neq X$, since $\cup_{j=1}^k Z_j \cap Tran = \emptyset$. Then $C_K = \overline{\cup_{\tau \in K} S_\tau} = S_\mu \sqcup S_\nu \neq X$ and note that $T|_{C_K}$ is not a transitive subsystem. By Lemma 3.7 (4) $G_K \subseteq X \setminus QW$. By (4.7) and Lemma 2.17 (1) $G_K \subseteq I_\Phi$. Note that $I_\Phi \cap I = I_\Phi$. Since T is convex-saturated, then

$$h_{top}(T, I_\Phi \cap I \setminus QW) \geq h_{top}(T, G_K) = \inf\{h_m \mid m \in K\} \geq \min\{h_\mu, h_\omega\} > h_{top} - \varepsilon.$$

□

5. GAP-SETS W.R.T. LYAPUNOV-LEVEL SET $R_{\Phi,a}$

In this section we discuss the entropy estimate on gap-sets w.r.t. Lyapunov-level set $R_{\Phi,a}$.

Proposition 5.1. *Suppose that T is single-saturated. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions with $I_\Phi \neq \emptyset$. Then for any real number $a \in \text{Int}(L_\Phi)$,*

- (1) $\{QR_{\text{erg}}, QR\}$ has full entropy gaps w.r.t. $R_{\Phi,a}$.
 - (1') $\{R, QR\}$ has full entropy gaps w.r.t. $R_{\Phi,a}$.
 - (2) If there is some invariant measure such that its support is not minimal, then $\{A \cup QR_{\text{erg}}, QR\}$ has full entropy gaps w.r.t. $R_{\Phi,a}$.
 - (2') If there is some invariant measure such that its support is not minimal, then $\{A \cup R, QR\}$ has full entropy gaps w.r.t. $R_{\Phi,a}$.
- If further there is an invariant measure with full support, then above gap-sets can be stated w.r.t. $R_{\Phi,a} \cap \text{Tran} \cap W \cap QR_d$.

Proof. The proof is similar as [82, Proposition 7.1] and [82, Proposition 7.2]. Here we omit the details. \square

Proposition 5.2. *Suppose that T is convex-saturated, there is some invariant measure with full support and $M_{\text{erg}}(T, X)$ is dense in $M(T, X)$. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions with $I_\Phi \neq \emptyset$. Then for any real number $a \in \text{Int}(L_\Phi)$,*

$$h_{\text{top}}(T, R_{\Phi,a} \cap \text{Tran} \cap W \setminus QR) = h_{\text{top}}(T, R_{\Phi,a}).$$

Proof. Fix $a \in \text{Int}(L_\Phi)$ and let $t = \sup\{h_\rho \mid \rho \in M(T, X) \text{ and } \chi_\Phi(\rho) = a\}$. By Proposition 2.18, we only need to show that

$$h_{\text{top}}(T, R_{\Phi,a} \cap \text{Tran} \cap W \setminus QR) \geq t.$$

Fix $\varepsilon > 0$. We need to construct two measures as follows, which are also useful to prove other propositions.

Lemma 5.3. *Suppose that there is some invariant measure with full support and $M_{\text{erg}}(T, X)$ is dense in $M(T, X)$. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions with $I_\Phi \neq \emptyset$. Then for any real number $a \in \text{Int}(L_\Phi)$, there are two different measures $\omega, \omega' \in M(T, X)$ ($\omega \neq \omega'$) such that*

$$\min\{h_\omega, h_{\omega'}\} > t - \varepsilon$$

and $\chi_\Phi(\omega) = \chi_\Phi(\omega') = a$, $S_\omega = S_{\omega'} = X$.

Proof. By Lemma 3.3 we can take three different ergodic measures of μ_i ($i = 0, 1, 2$) with support X such that

$$\chi_\Phi(\mu_0) < a < \chi_\Phi(\mu_1) < \chi_\Phi(\mu_2).$$

Then we can choose suitable $\theta_i \in (0, 1)$ ($i = 1, 2$) such that $\nu_i = \theta_i \mu_0 + (1 - \theta_i) \mu_i$ satisfy

$$\chi_\Phi(\nu_i) = a, i = 1, 2.$$

Remark that by ergodicity of μ_i , $\nu_1 \neq \nu_2$ and $S_{\nu_i} = S_{\mu_0} \cup S_{\mu_i} = X$, $i = 1, 2$.

By definition of t , we can take $\mu \in M(T, X)$ such that $\chi_\Phi(\mu) = a$ and $h_\mu > t - \varepsilon$. Then we can choose $0 < \theta < 1$ close to 1 such that $\omega = \theta \mu + (1 - \theta) \nu_1$, $\omega' = \theta \mu + (1 - \theta) \nu_2$ satisfy

$$h_\omega = \theta h_\mu + (1 - \theta) h_{\nu_1} \geq \theta h_\mu > t - \varepsilon,$$

$$h_{\omega'} = \theta h_\mu + (1 - \theta) h_{\nu_2} \geq \theta h_\mu > t - \varepsilon.$$

Remark that $\chi_\Phi(\omega) = \chi_\Phi(\omega') = a$, $S_\omega = S_{\omega'} = X$ and $\nu_1 \neq \nu_2$ implies $\omega \neq \omega'$. \square

Now we continue the proof of Proposition 5.2. Let $K = \{\tau \omega + (1 - \tau) \omega' \mid \tau \in [0, 1]\}$, then for any $m = \tau \omega + (1 - \tau) \omega' \in K$,

$$S_m = X, h_m \geq \min\{h_\omega, h_{\omega'}\} > t - \varepsilon, \chi_\Phi(m) = a.$$

By Lemma 3.7 (1) and Lemma 2.17 (2) $G_K \subseteq R_{\Phi,a} \cap Tran \cap W$. Since K is not a singleton, $G_K \subseteq I$. Since T is convex-saturated, then

$$h_{top}(T, R_{\Phi,a} \cap Tran \cap W \setminus QR) \geq h_{top}(T, G_K) = \inf\{h_m \mid m \in K\} > t - \varepsilon.$$

□

Proposition 5.4. *Suppose that T is a convex-saturated system with entropy-dense property, and there is some invariant measure with full support. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions with $I_\Phi \neq \emptyset$. Then for any real number $a \in Int(L_\Phi)$,*

$$h_{top}(T, R_{\Phi,a} \cap Tran \cap V \setminus W) = h_{top}(T, R_{\Phi,a}).$$

Compared with [82, Proposition 7.9], this proposition does not require density of periodic points and measures, just assuming a measure with full support instead.

Proof. Fix $a \in Int(L_\Phi)$ and let $t = \sup\{h_\rho \mid \rho \in M(T, X) \text{ and } \chi_\Phi \rho = a\}$. By Proposition 2.18, we only need to show that

$$h_{top}(T, R_{\Phi,a} \cap Tran \cap V \setminus W) \geq t.$$

Fix $\varepsilon \in (0, t)$. We need to construct a measure as follows, which is also useful to prove other propositions.

Lemma 5.5. *Suppose that T is a transitive system with entropy-dense property, and there is some invariant measure with full support. Then there is a measure $\omega \in M(T, X)$ such that*

$$h_\omega > t - \varepsilon \text{ and } \chi_\Phi(\omega) = a, S_\omega \subsetneq X.$$

Proof. Take a $\nu \in M(T, X)$ such that $h_\nu(f) > t - \frac{\varepsilon}{3}$ and $\chi_\Phi(\nu) = a$. From entropy-dense property, the set $B := \{\mu \in M_{erg}(T, X) \mid S_\mu \neq X\}$ is dense in $M(T, X)$. By Lemma 3.3, take two ergodic measures $\nu_i (i = 1, 2)$ with $S_{\nu_i} \neq X$ such that $b_1 := \chi_\Phi(\nu_1) > a > \chi_\Phi(\nu_2) =: b_2$. Let $\delta > 0$ small enough such that

$$\min\left\{\frac{b_1 - a}{b_1 - a + \delta}, \frac{a - b_2}{a - b_2 + \delta}\right\} > \frac{t - \varepsilon}{t - \frac{2\varepsilon}{3}}.$$

Then by entropy-dense property, we can take one ergodic measure μ close to ν enough (in weak* topology) such that $S_\mu \subsetneq X$ and

$$|\chi_\Phi(\mu) - a| = |\chi_\Phi(\mu) - \chi_\Phi(\nu)| < \delta, h_\mu(f) > t - \frac{2\varepsilon}{3}.$$

If $\chi_\Phi(\mu) = a$, then take $\omega = \mu$. Otherwise, $\chi_\Phi(\mu) \neq a$. Without loss of generality, we assume $\chi_\Phi(\mu) < a$. Take

$$\omega = \frac{b_1 - a}{b_1 - \chi_\Phi \mu} \mu + \left(1 - \frac{b_1 - a}{b_1 - \chi_\Phi \mu}\right) \nu_1.$$

Then $\chi_\Phi(\omega) = a$, $h_\omega(f) \geq \frac{b_1 - a}{b_1 - \chi_\Phi \mu} h_\mu(f) > \frac{b_1 - a}{b_1 - a + \delta} h_\mu(f) > t - \varepsilon$. Recall that T is transitive. Then any finite compact invariant sets $Z_j \subsetneq X$, $j = 1, 2, \dots, k$, $\cup_{j=1}^k Z_j \neq X$, since $\cup_{j=1}^k Z_j \cap Tran = \emptyset$. So $S_\omega = S_\mu \cup S_{\nu_1} \neq X$. □

Now we continue the proof of Proposition 5.4. By Lemma 3.8 we know T is transitive so that Lemma 5.5 applies. Recall that from entropy-dense property, the set $B := \{\mu \in M_{erg}(T, X) \mid S_\mu \neq X\}$ is dense in $M(T, X)$. Then one can construct ω' same as in Lemma 5.3 such that

$$h_{\omega'} > t - \varepsilon, \chi_\Phi(\omega') = a \text{ and } S_{\omega'} = X.$$

Since $S_\omega \subsetneq X = S_{\omega'}$, then $\omega \neq \omega'$.

Let $K = \{\tau\omega + (1-\tau)\omega' \mid \tau \in [0, 1]\}$, then for any $m = \tau\omega + (1-\tau)\omega' \in K \setminus \{\omega\}$, $S_m = X$ and for any $m = \tau\omega + (1-\tau)\omega' \in K$,

$$h_m \geq \min\{h_\omega, h_{\omega'}\} > t - \varepsilon, \quad \chi_\Phi(m) = a.$$

By Lemma 3.7 (2) and Lemma 2.17 (2)

$$G_K \subseteq R_{\Phi,a} \cap \text{Tran} \cap V \setminus W.$$

Since T is convex-saturated, then

$$h_{\text{top}}(T, R_{\Phi,a} \cap \text{Tran} \cap V \setminus W) \geq h_{\text{top}}(T, G_K) = \inf\{h_m \mid m \in K\} > t - \varepsilon.$$

□

Proposition 5.6. *Suppose that T is saturated system with entropy-dense property, and there is some invariant measure with full support. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions with $I_\Phi \neq \emptyset$. Then for any real number $a \in \text{Int}(L_\Phi)$,*

$$h_{\text{top}}(T, R_{\Phi,a} \cap \text{Tran} \cap QW \setminus V) = h_{\text{top}}(T, R_{\Phi,a}).$$

Compared with [82, Proposition 7.11], this proposition does not require density of periodic points and measures, just assuming a measure with full support instead.

Proof. Fix $a \in L_\Phi$ and let $t = \sup\{h_\rho \mid \rho \in M(T, X) \text{ and } \chi_\Phi(\rho) = a\}$. By Proposition 2.18, we only need to show that

$$h_{\text{top}}(T, R_{\Phi,a} \cap \text{Tran} \cap QW \setminus V) \geq t.$$

Fix $\varepsilon > 0$. Firstly, by assumption we can take same ω as in Lemma 5.5 such that $\chi_\Phi(\omega) = a$, $h_\omega(f) > t - \varepsilon$ and $S_\omega \neq X$.

From entropy-dense property, the set $B := \{\mu \in M_{\text{erg}}(T, X) \mid S_\mu \neq X\}$ is dense in $M(T, X)$. By Lemma 3.3, there are two ergodic measures m_i , $i = 1, 2$, with $S_{m_i} \neq X$ such that $\chi_\Phi(m_1) > a > \chi_\Phi(m_2)$. By Lemma 3.10, $\overline{\cup_{\omega \in B} S_\omega} = X$. Since X and $M(T, X)$ are compact metric spaces, we can take a sequence of measures $\{m_i\}_{i \geq 3}$ contained in B such that the set $\cup_{i \geq 3} S_{m_i}$ is dense in X .

Let

$$K_1 := \{m \mid \chi_\Phi(m) > a, m \in \{m_i\}_{i=1}^\infty\},$$

$$K_2 := \{m \mid \chi_\Phi(m) < a, m \in \{m_i\}_{i=1}^\infty\}$$

and

$$K_3 := \{m \mid \chi_\Phi(m) = a, m \in \{m_i\}_{i=1}^\infty\}.$$

Note that K_1 and K_2 are not empty, since $m_i \in K_i$, $i = 1, 2$. Remark that K_3 may be empty, finite or countable. Without loss of generality, we can assume $K_i = \{m_j^{(i)}\}_{j=1}^\infty$, $i = 1, 2, 3$. Then we can choose suitable $\theta_{j,k} \in (0, 1)$ such that $m_{j,k} = \theta_{j,k} m_j^{(1)} + (1 - \theta_{j,k}) m_k^{(2)}$ satisfies $\chi_\Phi(m_{j,k}) = a$. For any $n \geq 1$, let

$$l_n = \frac{\sum_{j+k=n} m_{j,k} + m_n^{(3)}}{n},$$

then $\chi_\Phi(l_n) = a$. Remark that every S_{l_n} is a union of the supports of finite measures without full support, and $\bigcup_{n \geq 1} S_{l_n} = \cup_{i \geq 1} S_{m_i}$ is dense in X . By Lemma 3.8 we know T is transitive. Then any finite compact invariant sets $Z_j \subsetneq X$, $j = 1, 2, \dots, k$, $\cup_{j=1}^k Z_j \neq X$, since $\cup_{j=1}^k Z_j \cap \text{Tran} = \emptyset$. So $S_{l_n} \neq X$.

Take an increasing sequence of $\{\theta_i \mid \theta_i \in (0, 1)\}_{i=1}^\infty$ convergent to 1 such that $h_{\omega_i} > t - \varepsilon$ where $\omega_i = \theta_i \omega + (1 - \theta_i) l_i$. Remark that $S_{\omega_i} = S_\omega \cup S_{l_i}$. In particular, for all i , $\chi_\Phi(\omega_i) = \chi_\Phi(\omega) = a$.

Now we consider

$$K = \{\omega\} \cup \bigcup_{i \geq 1} \{\tau\omega_i + (1-\tau)\omega_{i+1} \mid \tau \in [0, 1]\}.$$

Then K is nonempty connected compact subset of $M(T, X)$ because $\omega_i \rightarrow \omega$ in weak* topology. Note that $C_K = \overline{\bigcup_{\tau \in K} S_\tau} = \overline{\bigcup_i S_{m_i} \cup S_\omega} = X$ and for any $\omega \in K$, there is some i such that $S_\omega \subseteq S_\mu \cup S_{m_i} \cup S_{m_{i+1}} \neq X$. By Lemma 3.7 (3) and Lemma 2.17 (2) $G_K \subseteq R_{\Phi, a} \cap Tran \cap QW \setminus V$. Since T is saturated, then

$$h_{top}(T, R_{\Phi, a} \cap Tran \cap QW \setminus V) \geq h_{top}(T, G_K) = \inf_{\nu \in K} h_\nu = \min\{\inf_{i \geq 1} \{h_{\omega_i}\}, h_\omega\} \geq t - \varepsilon.$$

□

Proposition 5.7. *Suppose that T is a convex-saturated system with entropy-dense property, and there is some invariant measure with full support. Let $\Phi := \{\phi_n : X \rightarrow \mathbb{R}\}_{n \geq 1}$ be an asymptotically additive sequence of functions with $I_\Phi \neq \emptyset$. Then for any real number $a \in \text{Int}(L_\Phi)$,*

(1) *If T is transitively-convex-saturated, then*

$$h_{top}(T, R_{\Phi, a} \cap Tran \cap BR^* \cap I \setminus QW) = h_{top}(T, R_{\Phi, a}).$$

(2) *If $\{\mu \in M(T, X) \mid S_\mu \text{ is minimal}\}$ is dense in $M(T, X)$, then*

$$h_{top}(T, R_{\Phi, a} \cap I \setminus QW) = h_{top}(T, R_{\Phi, a}).$$

Compared with [82, Proposition 7.4], the second item of this proposition replaces requirements on periodic orbits by minimal sets.

Proof. Fix $a \in L_\Phi$ and let $t = \sup\{h_\rho \mid \rho \in M(T, X) \text{ and } \chi_\Phi(\rho) = a\}$. By Proposition 2.18, we only need to show that

$$h_{top}(T, R_{\Phi, a} \cap Tran \cap BR \cap I \setminus QW) \geq t.$$

Fix $\varepsilon > 0$. Firstly, by assumption we can take same ω as in Lemma 5.5 such that $\chi_\Phi(\omega) = a$, $h_\omega(f) > t - \varepsilon$ and $S_\omega \neq X$. By entropy-dense property, $M_{erg}(T, X)$ and $B := \{m \in M_{erg}(T, X) \mid S_m \neq X\}$ are dense subsets of $M(T, X)$. By Lemma 3.6, $Tran \subseteq BR$.

(1) By Lemma 3.10, $\overline{\bigcup_{m \in B} S_m} = X$. Thus we can take $m_1 \in B$ such that $X \neq S_{m_1} \setminus S_\omega \neq \emptyset$. If $\chi_\Phi(m_1) = a$, take $m = m_1$. Otherwise, without loss of generality, we assume $\chi_\Phi(m_1) < a$. By Lemma 3.3, if let $\mathcal{R} = B$, then $I_\Phi \neq \emptyset$ implies that we can take an ergodic measure m_2 with $S_{m_2} \neq X$ such that $a < \chi_\Phi(m_2)$. Take suitable $\alpha \in (0, 1)$ such that the measure $m = \alpha m_1 + (1 - \alpha)m_2$ satisfies that $\chi_\Phi(m) = \alpha$. Take $\theta \in (0, 1)$ close to 1 such that $\omega' = \theta\omega + (1 - \theta)m$ satisfies $h_{\omega'} \geq \theta h_\omega > t - \varepsilon$. Then ω' also satisfies that

$$(5.1) \quad \chi_\Phi(\omega) = \chi_\Phi(\omega') = a.$$

By Lemma 3.8 we know T is transitive. Then any finite compact invariant sets $Z_j \subsetneq X$, $j = 1, 2, \dots, k$, $\bigcup_{j=1}^k Z_j \neq X$, since $\bigcup_{j=1}^k Z_j \cap Tran = \emptyset$. Then $S_{\omega'} \cup S_\omega = S_\omega \cup S_{m_1} \cup S_{m_2} \neq X$. Remark that $S_{\omega'} \supseteq S_\omega \cup S_{m_1} \supsetneq S_\omega$ and thus $\omega \neq \omega'$.

Let $K = \{\tau\omega' + (1-\tau)\omega \mid \tau \in [0, 1]\}$. Note that $C_K = \overline{\bigcup_{\tau \in K} S_\tau} = S_{\omega'} \cup S_\omega \neq X$. By Lemma 3.7 (5) $G_K^T \subseteq Tran \setminus QW$. By (5.1) and Lemma 2.17 (2) $G_K^T \subseteq R_{\Phi, a}$. By Lemma 3.6 $Tran \subseteq BR^*$. Since K is not a singleton, $G_K \subseteq I$. Since T is transitively-convex-saturated, then

$$\begin{aligned} h_{top}(T, Tran \cap BR^* \cap R_{\Phi, a} \cap I \setminus QW) &= h_{top}(T, Tran \cap R_{\Phi, a} \cap I \setminus QW) \\ &\geq h_{top}(T, G_K^T) = \inf_{m \in K} h_m \geq \min\{h_{\omega'}, h_\omega\} > t - \varepsilon. \end{aligned}$$

(2) By Lemma 3.11, $\overline{A} = X$. Since $X \setminus S_\omega$ is nonempty and open, by density of almost periodic points

$$J := \{\nu \in M(T, X) \mid S_\nu \text{ is minimal, } S_\nu \setminus S_\omega \neq \emptyset\} \neq \emptyset.$$

If there is $m \in J$ such that $\chi_\Phi(m) = a$, take $\mu = m$. Otherwise, for any $m \in J$, $\chi_\Phi(m) \neq a$. Take one $\mu_1 \in J$. Then $\chi_\Phi(\mu_1) \neq a$. Without loss of generality, we assume $\chi_\Phi(\mu_1) < a$. By assumption and Lemma 3.3 we can take an invariant measure μ_2 with S_{μ_2} being minimal such that $\chi_\Phi(\mu_2) > a$. Then we can choose suitable $\theta \in (0, 1)$ such that $\mu = \theta\mu_1 + (1 - \theta)\mu_2$ satisfies $\chi_\Phi(\mu) = a$. Remark that $S_\mu \setminus S_\omega$ is composed of one minimal set or two minimal set containing the minimal set S_{μ_1} .

Take $\theta' \in (0, 1)$ close to 1 such that $\omega' = \theta'\omega + (1 - \theta')\mu$ satisfies $h_{\omega'} > t - \varepsilon$. Remark that $\chi_\Phi(\omega') = a$ and $S_{\omega'} \setminus S_\omega = S_\mu \setminus S_\omega \neq \emptyset$ and thus $\omega \neq \omega'$.

Let $K = \{\tau\omega + (1 - \tau)\omega' \mid \tau \in [0, 1]\}$, then for any $\tau \in K$,

$$h_\tau \geq \min\{h_\omega, h_{\omega'}\} > t - \varepsilon, \quad \chi_\Phi(\tau) = a.$$

By Lemma 3.8 we know T is transitive. Then any finite compact invariant sets $Z_j \subsetneq X$, $j = 1, 2, \dots, k$, $\cup_{j=1}^k Z_j \neq X$, since $\cup_{j=1}^k Z_j \cap \text{Tran} = \emptyset$. Then $C_K = \overline{\cup_{\tau \in K} S_\tau} = S_\omega \cup S_{\mu_1} \cup S_{\mu_2} \neq X$ and note that $T|_{C_K}$ is not a transitive subsystem. By Lemma 3.7 (4) $G_K \subseteq X \setminus QW$. By Lemma 2.17 (2) $G_K \subseteq R_{\Phi, a}$. Since K is not a singleton, $G_K \subseteq I$. Since T is convex-saturated, then

$$h_{\text{top}}(T, R_{\Phi, a} \cap I \setminus QW) \geq h_{\text{top}}(T, G_K) = \inf\{h_m \mid m \in K\} \geq \min\{h_\mu, h_\omega\} > t - \varepsilon.$$

□

6. PROOFS OF MAIN RESULTS

6.1. Proof of Theorem A. The estimate on gap-sets w.r.t. $R_{\Phi, a}$ and I_Φ can be deduced from above two sections.

Now we consider the gap-sets w.r.t. to R_Φ and X .

Step 1. Firstly we consider the case of $I_\Phi \neq \emptyset$. That is, we need to prove that

$$\{A \cup R, QR, W, V, QW, I\}$$

has full entropy gaps with respect to R_Φ .

Fix $\varepsilon > 0$. By Proposition 2.19, we can take a number $a \in \text{Int}(L_\Phi)$ such that

$$h_{\text{top}}(T, R_{\Phi, a}) > h_{\text{top}} - \varepsilon.$$

Recall that

$$R_\Phi = \bigsqcup_{b \in \mathbb{R}} R_{\Phi, b}.$$

So

$$h_{\text{top}}(T, R_\Phi \cap \xi) \geq h_{\text{top}}(T, R_{\Phi, a} \cap \xi) = h_{\text{top}}(T, R_{\Phi, a}) > h_{\text{top}} - \varepsilon,$$

where

$$\xi = \text{Tran} \cap QR \setminus (A \cup R), \text{Tran} \cap W \setminus QR, \text{Tran} \cap V \setminus W, \text{Tran} \cap QW \setminus V, I \setminus QW.$$

Remark that ξ can be chosen $I \cap \text{Tran} \cap BR \setminus QW$ when the system is transitively-convex-saturated. By arbitrariness of ε , we complete the proof of case $I_\Phi \neq \emptyset$.

Step 2. The case that I_Φ is empty. That is, $R_\Phi = X$.

Since $M(T, X)$ is not a singleton, we can take two invariant measures $\nu_1 \neq \nu_2$. Take an invariant measure μ with positive entropy and let $\mu_i = \frac{1}{2}(\mu + \nu_i)$, $i = 1, 2$. Then $\mu_1 \neq \mu_2$ and thus by weak* topology, there is some continuous function $\varphi : X \rightarrow \mathbb{R}$ such that $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$. Consider the sequence of additive functions (certainly also asymptotically additive) $\Psi := \{\phi_n := \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i)\}_{n \geq 1}$. Let $K =$

$\{t\mu_1 + (1-t)\mu_2 \mid t \in [0, 1]\}$. Since T is convex-saturated, one can get $h_{top}(T, I_\Psi) \geq h_{top}(T, G_K) = \min_{i=1,2} h_{\mu_i}(T) > 0$. So $I_\Psi \neq \emptyset$. By Step 1 for Ψ , the considered gap-sets has full entropy gaps with respect to R_Ψ . Since $R_\Psi \subseteq X = R_\Phi$, then considered gap-sets also have full entropy gaps w.r.t. R_Φ .

Step 3. Take Φ be a sequence of constant functions ($\equiv 1$). Then it is asymptotically additive and by Step 2, considered gap-sets also have full entropy gaps w.r.t. X .

Now we complete the proof of Theorem A. \square

6.2. Transitively saturated: Proof of Proposition A. By Lemma 3.13 the set $\{\omega \in M(T, X) \mid \mu \text{ is ergodic, } S_\omega \text{ is minimal}\}$ is dense in $M(T, X)$ and almost periodic set A is dense in X . So we only need to prove following Theorem 6.1 and Theorem 6.5 which imply Proposition A.

Theorem 6.1. *Let $T : X \rightarrow X$ be a continuous map of a compact metric space X with g -almost product property and uniform separation. Suppose that there is an invariant measure with full support. Then T is transitively-saturated.*

Proof. From [65, Theorem 4.1 (3)] $h_{top}(f, G_K) \leq \inf\{h_\mu(f) \mid \mu \in K\}$. Since G_K^T is contained in G_K , then $h_{top}(f, G_K^T) \leq h_{top}(f, G_K) \leq \inf\{h_\mu(f) \mid \mu \in K\}$.

The difficult part of the proof is to obtain a lower bound for $h_{top}(T, G_K^T)$. One can modify the construction in the proof of [65, Theorem 1.1] to obtain a closed subset $F \subseteq G_K$ such that the entropy of F close to $\inf\{h_\mu(f) \mid \mu \in K\}$ and simultaneously we can require that the chosen points in F is transitive. For convenience of readers, we give a precise construction as follows.

By connectedness of K , one has

Lemma 6.2. [65, Page 944] (or [25, Page 202]) *There exists a sequence $\{\alpha_1, \alpha_2, \dots\}$ in K such that*

$$\overline{\{\alpha_j : j \in \mathbb{N}, j > n\}} = K, \forall n \in \mathbb{N} \text{ and } \lim_{j \rightarrow \infty} d(\alpha_j, \alpha_{j+1}) = 0.$$

Let $\eta > 0$ and

$$h^* := \inf\{h_\mu(f) \mid \mu \in K\} - 2\eta, \quad H^* := \inf\{h_\mu(f) \mid \mu \in K\} - \eta.$$

Given this sequence of measures $\{\alpha_k\}$, we will construct a subset G such that for each $x \in G$, $\{\Upsilon_n(x)\}$ has the same limit-point set as the sequence $\{\alpha_k\}$, and $h_{top}(T, G) \geq h^*$. The construction of G is the core of the proof which is also used in the proof of Theorem 6.5 below.

By Lemma 2.6, we can find $\delta^* > 0, \epsilon^* > 0$ such that for $\mu \in M(T, X)$ and any neighborhood $F \subseteq M(X)$ of μ , there exists $n_{F, \mu, \eta}^*$, such that for $n \geq n_{F, \mu, \eta}^*$,

$$(6.1) \quad N(F; \delta^*, n, \epsilon^*) \geq 2^{n(h_\mu(f) - \eta)}.$$

Let $m : \mathbb{R}^+ \rightarrow \mathbb{N}$ be the nonincreasing function by g -almost product property. Let $\{\zeta_k\}$ and $\{\epsilon_k\}$ be two strictly decreasing sequences so that $\lim_{k \rightarrow \infty} \zeta_k = 0 = \lim_{k \rightarrow \infty} \epsilon_k$ with $\epsilon_1 < \epsilon^*$. By Lemma 3.13 almost periodic set A is dense in X . Thus for any fixed k there is a finite set $\Delta_k := \{x_1^k, x_2^k, \dots, x_{t_k}^k\} \subseteq A$ and $L_k \in \mathbb{N}$ such that Δ_k is ϵ_k -dense in X and for any $1 \leq i \leq t_k$, any $l \geq 1$, there is $n \in [l, l + L_k]$ such that $f^n(x_i^k) \in B(x_i^k, \epsilon_k)$. This implies that any $1 \leq i \leq t_k$,

$$(6.2) \quad \frac{\#\{0 \leq n \leq lL_k : d(T^n x_i^k, x_i^k) \leq \epsilon_k\}}{lL_k} \geq \frac{1}{L_k}.$$

Take l_k large enough such that

$$(6.3) \quad l_k L_k \geq m(\epsilon_k), \quad \frac{g(l_k L_k)}{l_k L_k} < \frac{1}{4L_k}.$$

We may assume the two sequences of $\{t_k\}, \{l_k\}, \{L_k\}$ are strictly increasing.

From (6.1) we get the existence of n_k and a $(\delta^*, n_k, \epsilon^*)$ -separated subset $\Gamma_k \subseteq X_{n_k, \mathcal{B}(\alpha_k, \zeta_k)}$ with

$$(6.4) \quad \#\Gamma_k \geq 2^{n_k H^*}$$

We may assume that n_k satisfies

$$(6.5) \quad n_k > m(\epsilon_k), \frac{t_k l_k L_k}{n_k} \leq \zeta_k, \delta^* n_k > 2g(n_k) + 1 \text{ and } \frac{g(n_k)}{n_k} \leq \epsilon_k$$

and

$$(6.6) \quad 2^{H^* n_k} \geq 2^{h^*(n_k + t_k l_k L_k)}.$$

We choose a strictly increasing $\{N_k\}$, with $N_k \in \mathbb{N}$, so that

$$(6.7) \quad n_{k+1} + t_{k+1} l_{k+1} L_{k+1} \leq \zeta_k \sum_{j=1}^k (n_j N_j + t_j l_j L_j)$$

and

$$(6.8) \quad \sum_{j=1}^{k-1} (n_j N_j + t_j l_j L_j) \leq \zeta_k \sum_{j=1}^k (n_j N_j + t_j l_j L_j).$$

Now we define the sequences $\{n'_j\}, \{\epsilon'_j\}$ and $\{\Gamma'_j\}$, by setting for

$$j = N_1 + N_2 + \cdots + N_{k-1} + t_1 + \cdots + t_{k-1} + q \text{ with } 1 \leq q \leq N_k,$$

$$n'_j := n_k, \epsilon'_j := \epsilon_k, \Gamma'_j := \Gamma_k$$

and for

$$j = N_1 + N_2 + \cdots + N_k + t_1 + \cdots + t_{k-1} + q \text{ with } 1 \leq q \leq t_k,$$

$$n'_j := l_k L_k, \epsilon'_j := \epsilon_k, \Gamma'_j := \{x_q^k\}.$$

Let

$$G_k := \bigcap_{j=1}^k \left(\bigcup_{x_j \in \Gamma'_j} T^{-M_j-1} B_{n'_j}(g; x_j, \epsilon'_j) \right) \text{ with } M_j := \sum_{l=1}^j n'_l.$$

Note that G_k is non-empty closed set. We can label each set obtained by developing this formula by the branches of a labeled tree of height k . A branch is labeled by (x_1, \dots, x_k) with $x_j \in \Gamma'_j$. Then Theorem 6.1 can be deduced by following lemma.

Lemma 6.3. *Let $\epsilon = \frac{1}{4}\epsilon^*$ and let*

$$G := \bigcap_{k \geq 1} G_k.$$

Then we have the following.

(1) *Let $x_j, y_j \in \Gamma'_j$ with $x_j \neq y_j$. If $x \in B_{n'_j}(g; x_j, \epsilon'_j)$ and $y \in B_{n'_j}(g; y_j, \epsilon'_j)$, then*

$$\max\{d(T^m x, T^m y) : m = 0, \dots, n_j - 1\} > 2\epsilon.$$

(2) *G is closed set that is the disjoint union of non-empty closed sets $G(x_1, x_2, \dots)$ Labeled by (x_1, x_2, \dots) with $x_j \in \Gamma'_j$. Two different sequences label two different sets.*

(3) *$G \subseteq G_K$.*

(4) *$h_{\text{top}}(T, G) \geq H^* - \eta = h^*$.*

(5) *$G \subseteq \text{Tran}$.*

Proof. Different with [65, Lemma 5.1], our new construction implies item (5). We can modify the proof of [65, Lemma 5.1] adaptable to our new construction and simultaneously the new construction guarantees item (5).

(1) Let $x \in B_{n'_j}(g; x_j, \epsilon'_j)$ and $y \in B_{n'_j}(g; y_j, \epsilon'_j)$. Since x_j and y_j are $(\delta^*, n'_j, \epsilon^*)$ -separated and (6.5) holds, there exists $m \in \Lambda_{n'_j}$ so that

$$d(T^m x_j, T^m y_j) > \epsilon^* = 4\epsilon, \quad d(T^m x_j, T^m x) \leq \epsilon'_j, \quad d(T^m y_j, T^m y) \leq \epsilon'_j.$$

However,

$$d(T^m x, T^m y) \geq d(T^m x_j, T^m y_j) - d(T^m x_j, T^m x) - d(T^m y_j, T^m y) > 2\epsilon.$$

(2) Note that G is the intersection of closed sets. Let (x_1, x_2, \dots) be a sequence with $x_j \in \Gamma'_j$. By the g -almost product property and compactness

$$\bigcap_{j \geq 1} T^{-M_{j-1}} B_{n'_j}(g; x_j, \epsilon'_j)$$

is nonempty and closed. By item (1) the two sets of $B_{n'_j}(g; x_j, \epsilon'_j)$ and $B_{n'_j}(g; y_j, \epsilon'_j)$ are disjoint when $x_j \neq y_j$. So two different sequences label two different sets.

(3) Define the stretched sequence $\{\alpha'_m\}$ by

$$\alpha'_m := \alpha_k \quad \text{if} \quad \sum_{j=1}^{k-1} (n_j N_j + t_j l_j L_j) + 1 \leq m \leq \sum_{j=1}^k (n_j N_j + t_j l_j L_j).$$

Then the sequence $\{\alpha'_m\}$ has the same limit-point set as the sequence of $\{\alpha_k\}$. If

$$\lim_{n \rightarrow \infty} d(\Upsilon_n(y), \alpha'_n) = 0,$$

then the two sequences $\{\Upsilon_n(y)\}, \{\alpha'_n\}$ have the same limit-point set. By (6.7) $\lim_{n \rightarrow \infty} \frac{M_{n+1}}{M_n} = 1$. So from the definition of $\{\alpha'_n\}$, we only need to prove that for any $y \in G$, one has

$$\lim_{l \rightarrow \infty} d(\Upsilon_{M_l}(y), \alpha'_{M_l}) = 0.$$

Assume that $\sum_{j=1}^k (n_j N_j + t_j l_j L_j) + 1 \leq M_l \leq \sum_{j=1}^{k+1} (n_j N_j + t_j l_j L_j)$. If $M_l \leq \sum_{j=1}^k (n_j N_j + t_j l_j L_j) + n_{k+1} N_{k+1}$, by Lemma 2.1 and (6.5)

$$d(\Upsilon_{M_l - \sum_{j=1}^k (n_j N_j + t_j l_j L_j)}(T^{\sum_{j=1}^k (n_j N_j + t_j l_j L_j)} y), \alpha_{k+1}) \leq \zeta_{k+1} + 2\epsilon_{k+1}.$$

Otherwise, $M_l > \sum_{j=1}^k (n_j N_j + t_j l_j L_j) + n_{k+1} N_{k+1}$, by Lemma 2.1, (6.5) and (2.1)

$$\begin{aligned} & d(\Upsilon_{M_l - \sum_{j=1}^k (n_j N_j + t_j l_j L_j)}(T^{\sum_{j=1}^k (n_j N_j + t_j l_j L_j)} y), \alpha_{k+1}) \\ & \leq \frac{n_{k+1} N_{k+1}}{M_l - \sum_{j=1}^k (n_j N_j + t_j l_j L_j)} d(\Upsilon_{n_{k+1} N_{k+1}}(T^{\sum_{j=1}^k (n_j N_j + t_j l_j L_j)} y), \alpha_{k+1}) \\ & \quad + \frac{M_l - \sum_{j=1}^k (n_j N_j + t_j l_j L_j) - n_{k+1} N_{k+1}}{M_l - \sum_{j=1}^k (n_j N_j + t_j l_j L_j)} \times 1 \\ & \leq 1 \times (\zeta_{k+1} + 2\epsilon_{k+1}) + \frac{t_{k+1} l_{k+1} L_{k+1}}{n_{k+1} N_{k+1}} \\ & \leq 2\zeta_{k+1} + 2\epsilon_{k+1}. \end{aligned}$$

By Lemma 2.1 and (6.5),

$$d(\Upsilon_{n_k N_k}(T^{\sum_{j=1}^{k-1} (n_j N_j + t_j l_j L_j)} y), \alpha_{k+1}) \leq \zeta_k + 2\epsilon_k + d(\alpha_k, \alpha_{k+1})$$

Thus, by (2.1), (6.8) and (6.5),

$$\begin{aligned}
& d(\Upsilon_{M_l}(y), \alpha_{k+1}) \\
& \leq \frac{\sum_{j=1}^{k-1} (n_j N_j + t_j l_j L_j)}{M_l} d(\Upsilon_{\sum_{j=1}^{k-1} (n_j N_j + t_j l_j L_j)}(y), \alpha'_{M_l}) \\
& \quad + \frac{n_k N_k}{M_l} d(\Upsilon_{n_k N_k}(T^{\sum_{j=1}^{k-1} (n_j N_j + t_j l_j L_j)} y), \alpha_{k+1}) + \frac{t_k l_k L_k}{M_l} \times 1 \\
& \quad + d(\Upsilon_{M_l - \sum_{j=1}^k (n_j N_j + t_j l_j L_j)}(T^{\sum_{j=1}^k (n_j N_j + t_j l_j L_j)} y), \alpha_{k+1}) \\
& \leq \frac{\sum_{j=1}^{k-1} (n_j N_j + t_j l_j L_j)}{\sum_{j=1}^k (n_j N_j + t_j l_j L_j)} \times 1 + 1 \times (\zeta_k + 2\epsilon_k + d(\alpha_k, \alpha_{k+1})) + \frac{t_k l_k L_k}{n_k} \\
& \quad + 2\zeta_{k+1} + 2\epsilon_{k+1} \\
& \leq \zeta_k + \zeta_k + 2\epsilon_k + d(\alpha_k, \alpha_{k+1}) + \zeta_k + 2\zeta_{k+1} + 2\epsilon_{k+1}.
\end{aligned}$$

Since $\zeta_k, \epsilon_k, d(\alpha_k, \alpha_{k+1})$ all converges to zero as k goes to zero, this proves item (3).

(4) As said in the proof of [65, Lemma 5.1, item 4] on Page 946, the details of the construction are unimportant and in fact Pfister and Sullivan proved that

Lemma 6.4. *If $\{n_p\}$ is a strictly increasing sequence of natural numbers such that $\lim_{p \rightarrow \infty} \frac{M_{n_{p+1}}}{M_{n_p}} = 1$ and $\#\Gamma'_{n_{p+1}} \times \#\Gamma'_{n_{p+2}} \cdots \times \#\Gamma'_{n_{p+1}} \geq 2^{h^*(M_{n_{p+1}} - M_{n_p})}$, then $h_{top}(T, G) \geq h^*$.*

For $k \geq 1$, $i = 0, 1, 2, \dots, N_k - 1$, let

$$n_{N_1 + \dots + N_{k-1} + i} := N_1 + \dots + N_{k-1} + t_1 + \dots + t_{k-1} + i$$

Then for any $p \geq 1$, there is some k so that $N_1 + \dots + N_{k-1} + t_1 + \dots + t_{k-1} \leq n_p \leq N_1 + \dots + N_{k-1} + t_1 + \dots + t_{k-1} + N_k - 1$, by (6.7)

$$\begin{aligned}
1 & \leq \frac{M_{n_{p+1}}}{M_{n_p}} \leq \frac{M_{n_p} + \max\{n_k, n_k + t_k l_k L_k\}}{M_{n_p}} = 1 + \frac{n_k + t_k l_k L_k}{M_{n_p}} \\
& \leq 1 + \frac{n_k + t_k l_k L_k}{\sum_{j=1}^{k-1} (n_j N_j + t_j l_j L_j)} \leq 1 + \zeta_k.
\end{aligned}$$

By (6.6)

$$\begin{aligned}
& \#\Gamma'_{n_{p+1}} \times \#\Gamma'_{n_{p+2}} \cdots \times \#\Gamma'_{n_{p+1}-1} \times \#\Gamma'_{n_{p+1}} \\
& = \#\Gamma'_k \geq 2^{H^* n_k} \geq 2^{h^*(n_k + t_k l_k L_k)} \geq 2^{h^*(M_{n_{p+1}} - M_{n_p})}.
\end{aligned}$$

By Lemma 6.4 we finish the proof of item (4).

(5) Fix $x \in G$. By construction, for any fixed $k \geq 1$, there is $a = a_k$ such that for any $j = 1, \dots, t_k$, there is $\Lambda^j \subseteq \Lambda_{l_k L_k}$

$$\max\{d(T^{a+l+(j-1)l_k L_k} x, T^l x_j^k) \mid l \in \Lambda^j\} \leq \epsilon_k.$$

By (6.3)

$$\frac{\#\Lambda^j}{l_k L_k} \geq 1 - \frac{g(l_k L_k)}{l_k L_k} \geq 1 - \frac{1}{4L_k}.$$

Together with (6.2) we get that for any $j = 1, \dots, t_k$ there is $p_j \in [0, l_k L_k - 1]$ such that

$$d(T^{a+p_j+(j-1)l_k L_k} x, T^{p_j} x_j^k) \leq \epsilon_k \text{ and } d(x_j^k, T^{p_j} x_j^k) \leq \epsilon_k.$$

This implies $d(T^{a+p_j+(j-1)l_k L_k} x, x_j^k) \leq 2\epsilon_k$ so that the orbit of x is $3\epsilon_k$ -dense in X . By arbitrariness of k , one has $x \in Tran$. \square

If we only have g -almost product property but do not know the property of uniform separation, then one still has following characterization.

Theorem 6.5. *Let $T : X \rightarrow X$ be a continuous map of a compact metric space X with g -almost product property property. Suppose that there is an invariant measure with full support. Then T is locally-transitively-convex-saturated and in particular locally-transitively-single-saturated.*

Proof. Similar as the construction of transitive points in Theorem 6.1, one can adapt the proof of [27, Theorem 1.5] (or [65, Theorem 1.2]) to complete the proof for which ergodic decomposition theorem replaces the role of uniform separation. Here we omit the details. \square

6.3. Comments on non-recurrence. Recently, Dong and Tian proved the existence of non-recurrent points, divided them into different types and they all carry full topological entropy, provided that the dynamical system is transitive or mixing, and is expansive with shadowing in [28]. These assumptions are stronger than uniform separation and g -almost product property, since mixing plus shadowing imply specification and expansiveness is stronger than uniform separation. A stronger saturated property is proved in [28] such that the topological entropy of various different types of non-recurrent points are studied. However, if one only assume uniform separation and g -almost product property, it is still unknown how to construct different types of non-recurrent points.

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